# Local Cauchy problem for the nonlinear Dirac and Dirac-Klein-Gordon equations on Kerr spacetime

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*Abstract.* - We prove the local existence of solutions of nonlinear Dirac and Dirac-Klein-Gordon equations in Kerr metric with regular Cauchy initial datas.

*Key Words.* - ADM decomposition, Black-hole, Dirac-Klein-Gordon equation, nonlinear Dirac equation, Kerr metric, General Relativity.

# 1 Introduction.

This paper deals with the Cauchy problem for non-linear Dirac and Dirac-Klein-Gordon equations in Kerr space-time. This curved space-time is one of physical relevant space-time solution of Einstein equations in vacuum. More precisely in Boyer-Lindquist coordinates on $\mathcal{M} := \mathbb{R}_t \times \mathbb{R}_r \times S_\omega$, we have :

$$\begin{aligned}
&g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{4aMr\sin^2\theta}{\rho^2}dtd\varphi - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{\sigma^2}{\rho^2}\sin^2\theta d\varphi^2,\\
&\sigma^2 := (r^2+a^2)\rho^2 + 2Mra^2\sin^2\theta, \qquad \Delta := r^2 - 2Mr + a^2,\\
&\rho^2 := r^2 + a^2\cos^2\theta.
\end{aligned} \tag{1}$$

The manifold $(\mathcal{M}, g)$ describes a rotating uncharged black hole where $M$ is its mass and $a$ is its angular momentum per unit mass. Briefly, we remark that we have two types of singularities: a true curvature singularity, the space of points $\{\rho; \rho^2 = 0\}$, and the coordinates singularities, the sphere(s) where $\Delta$ vanishes. This last property defines the horizon of the black-hole. Roughly speaking the sphere-horizon are the regions for which an observer does not cross and comes back them without a speed greater than the light. Finally the number of real roots of $\Delta$ ($\leq 2$) define three types of Kerr black hole :
- $\Delta$ has no real root *i.e.* for $|a| > M$, there are no horizon and the ring $\{\rho; \rho^2 = 0\}$ is a naked singularity.
- $\Delta$ has double root *i.e.* for $|a| = M$, $\{r = M\}$ is the only horizon, this is the extreme Kerr space-time.
- For $0 < |a| < M$, $\Delta$ has two real roots,

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{2}$$

so there are two horizons $\{r = r_-\}$ and $\{r = r_+\}$. This is the slow Kerr space-time.

In the sequel we consider this last type of Kerr black-hole. Hence the two horizons define three regions of the space-time. The block III ($\mathcal{B}_{III}$), $\{r < r_-\}$ contains the ring singularity and a time machine. The block II ($\mathcal{B}_{II}$), $\{r_- < r < r_+\}$, is a dynamic region where an inertial observer is dragged toward the horizon $\{r = r_-\}$. The block I ($\mathcal{B}_I$), $\{r > r_+\}$ is the exterior of the black-hole. Moreover, $\mathcal{B}_I$ is not stationary *i.e.* the killing vector field $K := \partial/\partial t$ is not timelike in all block I. The region $\mathcal{E} \subset \mathcal{B}_I$ where $K$ is spacelike in $\mathcal{B}_I$ ($g_{tt} < 0$) is called the ergosphere:

$$\mathcal{E} := \left\{(t,r,\theta,\varphi) : \; r_+ < r < M + \sqrt{M^2 - a^2\cos^2\theta}\right\}.$$

We study on $(\mathcal{B}_I, g)$ the solutions of the nonlinear Dirac equation (NLD) :

$$i\gamma^\mu \nabla_\mu \Psi - m_d \Psi = k(\Psi^* \mathcal{V}_1 \Psi)\Psi, \tag{3}$$

where $m_d > 0$ is the mass of the spin 1/2 field, $\mathcal{V}_1$ a matrix and $k \in L^\infty(\mathcal{B}_I)$. $\gamma^\mu$ are the Dirac matrices. We also study the Dirac-Klein-Gordon system (DKG) :

$$\begin{aligned} i\gamma^\mu \nabla_\mu \Psi - m_d \Psi &= \Phi \mathcal{V}_2 \Psi, \\ \Box_g \Phi + m_{kg}^2 \Phi &= \Psi^* \mathcal{V}_3 \Psi \end{aligned} \tag{4}$$

where $m_{kg} > 0$ is the mass of the spin 0 field, $\mathcal{V}_2$, $\mathcal{V}_3$ two matrices and $\Box_g$ the laplacian for the lorentzian metric $g$.

The difficulties of these studies on this type of space-time are principally due to the fact that we have to work with a curved manifold with less symmetries than flat space-time. Moreover in $\mathcal{B}_I$ of Kerr black-hole and for the spin 0 fields (Wave, Klein-Gordon equations) the phenomenon of the super-radiance takes place. It is a consequence of the absence of globally defined timelike Killing vector field that implies the nonexistence of positive-definite conserved quantities useful to define a functional framework to study the field equation. The super-radiance is the analogue for spin 0 fields of the Penrose experience of extraction of energy from the ergosphere. On the other hand the spin 1/2 field always possess a conserved current inducing a positive-definite inner product. Therefore the super-radiance seems not to be a real problem to prove the existence of the solutions of (3) unlike of (4) since the system consists in the Klein-Gordon field of spin 0.

The first paper about Cauchy problem for a non linear field equation in the black-hole background concerns the Klein-Gordon equation [32]. Later, the author extended in [34] his first work for the non linear Klein-Gordon equation in Schwarzschild metric ($a = 0$ in (1)) to the Kerr metric. To overcome the problem of the super-radiance, he used the geometrical 3+1 decomposition (ADM) of the Block I. It consists in adopt the coordinates of fixed observer with the respect to infinity. Indeed, we recall that the exterior of the black-hole $\mathcal{B}_I$ is dynamic, *i.e* an inertial observer turns with the black-hole. Using the ADM decomposition we have outside the black-hole :

$$\mathcal{B}_I = \mathbb{R}_\tau \times \Sigma, \quad g = N^2 d\tau^2 - h(\tau) \tag{5}$$

where $N$ is called the lapse function and $h$ is the spacelike metric. We prove that $N \to 0$ at the horizon and that $h(\tau)$ is equivalent uniform in space and locally uniform in time to the euclidean metric outside a unit closed ball in $\mathbb{R}^3$ ($\Sigma \cong \mathbb{R}^3 \setminus \bar{B}(0,1)$). In this framework the Klein-Gordon equation (but also our equations (3) and (4)) have the form of an evolution problem on natural Hilbert space. The norm of this Hilbert space is not *a priori* conserved but it is controlled by an energy estimate. The author of [34] proves the global existence and the uniqueness for weakly regular initial data (energy data $H^1 \oplus L^2$). Indeed this regularity is sufficient to give sense of an equivalent integral formulation of the non linear Cauchy problem since we have a cubic non linearity and the Sobolev embedding $H^1 \hookrightarrow L^6$. Moreover the globalization is obtained thanks to an energy estimate.

For our problems, the charge spaces are $L^2$ for (NLD) and $L^2 \oplus H^1 \oplus L^2$ for (DKG). These regularities do not allow us to control the non linear terms of the equations. Hence, we take the more regular data for our study: $H^2$ for (NLD) and $H^2 \oplus H^2 \oplus H^1$ for (DKG). The important property that $h(t)$ is equivalent to the euclidean metric outside a unit closed ball gives, thanks to the flat Sobolev embedding in $\mathbb{R}^3$ $H^2 \hookrightarrow L^\infty$, the same embedding for the curved space. This allows us to control the non linear terms. The global existence is much more difficult since for our equations, we do not have the conservation or control of the norm $H^2$ for (NLD) or $H^2 \oplus H^2 \oplus H^1$ for (DKG). Usually for (3) and (4) in the flat space-time, we obtain the global existence for the sufficiently small initial data in often using the sharper estimations. See this as well as possible exhaustive list of papers: For Dirac semilinear equation: [35], [36], [19]. For Non Linear Dirac equation [30], [27], [28], [37], [31], [15], [4], [5], [27], [16], [17], [14], [13], [12], [11]. For Dirac-Klein-Gordon [20], [29], [26], [8], [7], [6], [40], [18], [3], [2], [1], [39], [38], [10], [9]. Obviously the sharp estimates in curved space-time (like Strichartz) are a problem of great complexity, and in ours case there are not proved.

The paper is organized as follows. The second section concerns the $3+1$ decomposition. In the third and fourth we treat respectively of Cauchy problem for Dirac and Dirac-Klein-Gordon equation.

## 2 The ADM $3+1$ decomposition of the Kerr Block I

Usually, to describe the Kerr space-time we adopt the Boyer-Lindquist coordinates $(t, r, \omega) \in \mathbb{R}_t \times \mathbb{R}_r^+ \times S_\omega^2$. We have a time function $t$ globally defined in Block I, *i.e.* $\nabla t$ is a timelike future oriented vector field in $\mathcal{B}_I$. This function provides a foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$ of $\mathcal{B}_I$ by its level Cauchy hypersurfaces. Moreover, $K = a.\partial/\partial t$, $a \in \mathbb{R}$, is the only Killing vector field timelike near spacelike infinity. It fixes the product structure $\mathcal{B}_I := \mathbb{R}_t \times \Sigma$ *i.e.* the identification of points of $\Sigma_t$ along the integral lines of $K$. We remark that $\partial/\partial t$ is not timelike everywhere in $\mathcal{B}_I$. Indeed, $g_{tt} \leq 0$ if $(t, r, \omega) \in \mathcal{E}$ and $g_{tt} \geq 0$ if $(t, r, \omega) \in \mathcal{B}_I \backslash \mathcal{E}$. Hence, there exists no global timelike Killing vector field in $\mathcal{B}_I$. Therefore, this space time is not stationary. We know that the function $t$ of the Boyer-Lindquist coordinates is a time function. Hence $t$ can be used as time parameter in an evolution equation on $\mathcal{B}_I = \mathbb{R}_t \times \Sigma$,

$$\Sigma = ]r_+, \infty[\times S_\omega^2. \tag{6}$$

Since the metric $g$ is time independent in the Boyer Lindquist coordinates ($\partial/\partial t$ is a Killing vector field), then the coefficients of the field equation are also time independent.

Now, we point out the difficulties linked to the Boyer-Lindquist coordinates to study a Cauchy problem for equations (3) and (4). These difficulties are mainly due to the non globally timelike definition of the Killing vector field $\partial/\partial t$. Indeed, we consider $T_{\mu\nu}$ the stress-energy-momentum tensor for the following scalar field $u$ such that

$$\Box_g u + m^2 u = 0, \qquad \nabla^\mu T_{\mu\nu} = 0. \tag{7}$$

Since $\partial/\partial t$ is a Killing vector then the 1-form $T_{\mu 0}dx^\mu$ is closed. Now, we denote by $\mathbf{T}$ the unit future oriented vector field which is normal to $\Sigma_t$ such that

$$\mathbf{T}^\mu \frac{\partial}{\partial x^\mu} = \sqrt{\frac{\sigma^2}{\Delta\rho^2}} \left( \frac{\partial}{\partial t} + \frac{2aMr}{\sigma^2} \frac{\partial}{\partial \varphi} \right). \tag{8}$$

Then the energy of the field measured by an observer (static at infinity) whose 4-velocity vector is $\partial/\partial t$ is given by

$$E(u,t) := \int_{\Sigma_t} \mathbf{T}^\mu T_{\mu 0} dVol, \quad dVol = \sqrt{\frac{\rho^2\sigma^2}{\Delta}} dr d\omega \tag{9}$$

where $dVol$ is the volume measure on $\Sigma$. Hence,

$$E(u,t) = \int_\Sigma \left( |\partial_t u(t)|^2 + \frac{\Delta^2}{\sigma^2}|\partial_r u(t)|^2 + \frac{\Delta}{\sigma^2}|\partial_\theta u(t)|^2 + \frac{\rho^2 - 2Mr}{\sigma^2 \sin^2\theta}|\partial_\varphi u(t)|^2 + \frac{\Delta\rho^2 m^2}{\sigma^2}|u(t)|^2 \right) \frac{\sigma^2}{\Delta} dr d\omega \tag{10}$$

Clearly, the fourth term is positive outside the ergosphere and negative inside. Hence the energy is not positive definite in all $\mathcal{B}_I$. This property allows superradiance to take place outside the black hole. Hence we do not choose this energy space to study the Cauchy problem with a system which contains a spin 0 field as (4). Then, the study is more difficult but not impossible, see for example the strategy used in [24]. Roughly speaking, it consists in finding a new energy norm positive definite such that its growth is controlled by an energy estimate. Finally in the sequel, we adopt this approach but the choice of the new energy space is naturally given by the geometry thanks to the $3+1$ decomposition. Although in the case of the spin $1/2$ the phenomenon of superradiance does not take place, we show in the following paragraph that the $3+1$ decomposition is nevertheless useful for this field.

The space-time $(\mathcal{B}_I, g)$ is globally hyperbolic. This means that there exists a time function $t$ globally defined on $\mathcal{B}_I$ (providing of foliation of $\mathcal{B}_I$ by the hypersurfaces $\Sigma_t$) and that any points of $\mathcal{B}_I$ can be reached from $\Sigma_{t_0}$ along a non-spacelike curve (see Geroch [22]). Each $\Sigma_t$ are homeomorphic to a given 3-manifold $\Sigma$. Now, we choose $\mathbf{T}$ the unit timelike oriented vector normal to $\Sigma_t$ defined in (8) to fixed the product structure $\mathcal{B}_I = \mathbb{R} \times \Sigma$ *i.e.* the points on different hypersurfaces $\Sigma_t$ are identified along the integral lines of $\mathbf{T}$. This construction induces an explicit system of coordinates that is referred to as the point of view of locally non rotating observers :

$$\tau = t, \quad R = r, \quad \Theta = \theta, \quad \Phi = \varphi - t\alpha, \quad \alpha := -\frac{g_{t\varphi}}{g_{\phi\phi}} = \frac{2aMr}{\sigma^2}. \tag{11}$$

Now, we decompose the metric $g$ as the sum of its orthogonal projection along $\mathbf{T}$ and $(\mathbf{T})^\perp = T_p\Sigma_t$ :

$$g_{\mu\nu}dx^\mu dx^\nu = N^2 d\tau^2 - h(\tau), \qquad N := \sqrt{g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}}} = \sqrt{\frac{\Delta\rho^2}{\sigma^2}} \tag{12}$$

where $N$ is the lapse function, and

$$h(\tau) = -g_{rr}dR^2 - g_{\theta\theta}d\Theta^2 - g_{\varphi\varphi}\left(d\Phi + \tau\frac{\partial\alpha}{\partial R}dR + \tau\frac{\partial\alpha}{\partial\Theta}d\Theta\right)^2. \tag{13}$$

The definition of $g$ in these new coordinates leads to simple form of hyperbolic evolution equation without crossed terms depending on $t$ and $\varphi$ since $g_{t\varphi} = 0$ (unlike to the Boyer-Lindquist coordinates form for $g$). This is the main motivation to use the $3+1$ decomposition for spin $1/2$ fields. But we remark that the metric is not time independent anymore. Henceforth, the evolution equation is defined with the help of time dependent hamiltonian. The following proposition state that the dependence on $t$ is rather nice (see [33]) :

**Proposition 2.1**
*- $\Sigma_\tau = (\Sigma, h(\tau))$ is a $C^\infty$-Riemannian manifold for all $\tau \in \mathbb{R}$ with smooth boundary $\partial\Sigma = \{r_+\} \times S^2_\omega$.*
*- The lapse function $N$ is strictly positive on $\Sigma$, vanishes on $\partial\Sigma$, is independent of $\tau$, $C^\infty$ and uniformly bounded on $\bar{\Sigma}$ as well as all its derivatives.*
*- $h_{\mu\nu} \in C^\infty(\mathbb{R}_\tau; C^\infty_b(\Sigma; T_{\mu\nu}\mathcal{M}))$, $h^{\mu\nu} \in C^\infty(\mathbb{R}_\tau; C^\infty_b(\Sigma; T^{\mu\nu}\mathcal{M}))$. All slice $\Sigma_\tau$ have the same geometry ($g$ is independent of $t$ in Boyer-Lindquist coordinates) and $h(\tau)$ is obtained from $h(0)$ by a rotation around the axis of the black hole whose angle is $\tau\alpha$ with $\alpha$ defined in (11) :*

$$h(0) = \frac{\rho^2}{R^2}du^2 + \frac{\rho^2}{(1+u)^2}(1+u)^2 d\Theta^2 + \frac{(R^2+a^2)\rho^2 + 2MRa^2\sin^2\Theta}{\rho^2(1+u)^2}(1+u)^2\sin^2\Theta d\Phi^2 \tag{14}$$

*with $u$ the $h$-distance to the horizon such that*

$$u : [r_+, +\infty[_R \longmapsto \mathbb{R}, \qquad u(R) := \int_{r_+}^{R} \frac{s}{\sqrt{\Delta}}ds. \tag{15}$$

*Hence, we remark that $h(\tau)$ is equivalent to the Euclidian metric on $\mathbb{R}^3 \setminus \bar{B}(0,1)$:*

$$du^2 + (1+u)^2 d\Theta^2 + (1+u)^2\sin^2\Theta d\Phi^2. \tag{16}$$

*- $|h|$, the determinant of the metric $h(\tau)$, is independent of $\tau$ and $|g| := |det\ g| = -N^2|h|$.*

## 3 Local Cauchy problem for non linear Dirac equation outside a Kerr Black hole

In this section we study the local Cauchy problem for the non linear Dirac equation (3). We use the $3+1$ decomposition to defined an evolution problem in $\mathcal{B}_I$. A result of Nicolas [33] gives the existence of a propagator for a solution of the linear part of (3). For the non linear equation, a Duhamel formula and a Sobolev embedding $H^2 \hookrightarrow L^\infty$ are useful to obtain the result.

We describe more precisely the Dirac equation in $3+1$ decomposition framework. Since $(\mathcal{B}_I, g)$ is a globally hyperbolic spacetime then it admits a spin structure ([21, 23]). We denote by $\mathbb{S}$ the bundle over $\mathcal{B}_I$ of negative spinors and by $\bar{\mathbb{S}}$ the bundle of positive spinors *i.e.* the complex structure in $\mathbb{S}$ simply replaced by its opposite. We also respectively denote by the $\mathbb{S}^*$ and $\bar{\mathbb{S}}^*$ the dual of $\mathbb{S}$ and $\bar{\mathbb{S}}$. Finally the complexified tangent bundle to $\mathcal{B}_I$ is recovered as the tensor product of $\mathbb{S}$ and $\bar{\mathbb{S}}$,

$$T\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S} \otimes \bar{\mathbb{S}}, \quad T^*\mathcal{B}_I \otimes \mathbb{C} = \mathbb{S}^* \otimes \bar{\mathbb{S}}^*. \tag{17}$$

We define the Dirac equation on the block I. The bundle of Dirac spinors on $\mathcal{B}_I$ is described as

$$\mathbb{S}_D := \mathbb{S}^* \oplus \bar{\mathbb{S}} \tag{18}$$

On space time $\mathcal{B}_I$, we choose a local orthogonal Lorentz frame $\{e_0, e_1, e_2, e_3\}$ such that

$$g(e_0, e_0) = 1, \; g(e_a, e_a) = -1, \; a = 1, 2, 3 \quad g(e_a, e_b) = 0, \;\; a \neq b. \tag{19}$$

Obviously in the $3+1$ decomposition framework, we choose for the basis $\{e_0, e_1, e_2, e_3\}$

$$e_0{}^a := \frac{1}{\sqrt{2}} T^a, \qquad e_1, e_2, e_3 \in T\Sigma_t. \tag{20}$$

Hence, we define the Dirac operator on $\mathcal{B}_I$ by

$$\boldsymbol{D} := \sum_{a=0}^{3} e_a . \nabla_{e_a} \tag{21}$$

where $\nabla_{e_a}$ is the directional convariante derivative along $e_a$ and $e_a.$ the Clifford product by the vector $e_a$. More presicely by a choice of spin-frame or Newman-Penrose tetrad, the Clifford multiplication of a Dirac spinor $\Psi \in \mathbb{S}_D$ by $e_a$ is described as the multiplication by a Dirac matrices $\gamma^a$ satisfying the following relation

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}_4, \quad \mu, \nu = 0, 1, 2, 3, 4. \tag{22}$$

The Dirac equation for a spin 1/2 particle with mass $m_d > 0$ takes the following form with Einstein notation:

$$i\gamma^\mu \nabla_{e_\mu} \Psi - m_d \Psi = (\boldsymbol{D} - m_d)\Psi = 0. \tag{23}$$

From (20) and (23), we write the Dirac equation as an evolution system :

$$ie_0 . \nabla_{e_0} \Psi = -i \sum_{a=1}^{3} e_a . \nabla_{e_a} \Psi + m_d \Psi \tag{24}$$

or

$$\nabla_{e_0} \Psi = -\sum_{a=1}^{3} e_0 . e_a . \nabla_{e_a} \Psi - i m_d \Psi. \tag{25}$$

We introduce the Dirac-Witten operator $\boldsymbol{D}_w(\tau)$ on $\Sigma_\tau$ (extrinsic geometry) such that

$$\boldsymbol{D}_w(\tau) = \sum_{a=1}^{3} e_a.\nabla_{e_a}. \tag{26}$$

We also define $\boldsymbol{D}_\Sigma(\tau)$ the Dirac operator associated to the Levi-Civita connection on $(\Sigma_\tau, h(\tau))$ such that

$$\boldsymbol{D}_w(\tau) = \boldsymbol{D}_\Sigma(\tau) + \frac{1}{2\sqrt{2}} K e_0 \tag{27}$$

where $K$ is the $\sqrt{2}$ times the trace of the extrinsic curvature. This operator is symmetric on $C_0^\infty(\Sigma_t, \mathbb{S}_D)$ for the inner product

$$\langle \Psi, \Phi \rangle_{L^2_{\Sigma_\tau}} := \int_\Sigma \langle \Psi, \Phi \rangle \, dVol_{h_\tau}, \tag{28}$$

where

$$\langle \Psi, \Phi \rangle := \Psi_1 \bar{\Phi}_1 + \Psi_2 \bar{\Phi}_2 + \Psi_3 \bar{\Phi}_3 + \Psi_4 \bar{\Phi}_4 \tag{29}$$

when we choose a spin-frames adapted to the foliation $\Sigma_\tau$. According to Proposition 2.1, this inner product is en fact $\tau$-independent since the determinent of $g$ is also $\tau$-independent. Finally, we write the Dirac equation in the following form, *i.e.* as a first order symmetric hyperbolic system on $\Sigma$:

$$\nabla_{e_0} \Psi = -e_0.\boldsymbol{D}_\Sigma(\tau)\Psi - \frac{1}{2\sqrt{2}} K\Psi - i m_d \Psi, \tag{30}$$

and the operator

$$\boldsymbol{D}_D := e_0.\boldsymbol{D}_\Sigma(\tau), \qquad e_0 = \frac{1}{\sqrt{2}} \mathbf{T}^\mu \partial_\mu \tag{31}$$

being formally skew-adjoint on $L^2(\Sigma; \mathbb{S}_D)$. By choosing a adapted Newman-Penrose tetrad (a spin frame adapted to foliation $\Sigma_\tau$) equation (30) becomes (see Appendix A in [33]):

$$\frac{\partial \Psi}{\partial \tau} = \mathcal{A}_D(\tau)\Psi, \quad \mathcal{A}_D(\tau) := -\frac{N}{\sqrt{2}} \left( \boldsymbol{D}_\Sigma(\tau) + \frac{1}{2\sqrt{2}} K + i m_d \gamma^0 + B(\tau) \right) \tag{32}$$

where $B$ is a matrix containing the connections terms of $\nabla_{a_0}$, $N$ the lapse function and

$$\gamma^0 = i \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}. \tag{33}$$

We introduce the functional framework. On $\Sigma_\tau$, we define the Sobolev space $H_0^k(\Sigma_\tau; \mathbb{S}_D)$ as the completion of $C_0^\infty(\Sigma; \mathbb{S}_D)$ for the norm :

$$\|\Psi\|_{H_0^k(\Sigma_\tau)} = \left( \sum_{p=0}^{k} \int_\Sigma \langle (D_\tau)^p \Psi, (D_\tau)^p \Psi \rangle \, dVol_h \right)^{1/2} \tag{34}$$

where $D_\tau$ is the Levi-Civita connection on $(\Sigma, h(\tau))$. According Proposition 2.1, we remark that for all $\tau \in \mathbb{R}$, $dVol_h = dVol_{h(\tau)}$ and the norms in $H_0^k(\Sigma_\tau; \mathbb{S}_D)$ and $H_0^k(\Sigma_{\tau_0}; \mathbb{S}_D)$ are locally uniformly in time equivalent (the constants in the norms estimates are time dependent and locally bounded in time). Hence, we simply denote by $H_0^k(\Sigma, \mathbb{S}_D)$ the Sobolev space associated with the norm

$$\|.\|_{(k)} := \|.\|_{H_0^k(\Sigma_{\tau_0})}$$

and the metric $h(\tau_0)$.

**Proposition 3.1**
*We define the following norm for all* $\tau \in \mathbb{R}$*:*

$$\|\Psi\|_{k,\tau} := \left( \sum_{p=0}^{k} \int_{\Sigma} \langle (\boldsymbol{D}_\Sigma)^p(\tau)\Psi, (\boldsymbol{D}_\Sigma)^p(\tau)\Psi \rangle \, dVol_h \right)^{1/2}, \quad \forall \Psi \in H_0^k(\Sigma;\mathbb{S}_D). \tag{35}$$

*Then the norm* $\|.\|_{(k)}$ *and* $\|.\|_{k,\tau}$ *are locally uniformly in time equivalent on* $H_0^k(\Sigma;\mathbb{S}_D)$*.*

This proposition is a consequence of Proposition 2.1 and the Bochner-Lichnerowicz-Weitzenböck formula:

$$(\boldsymbol{D}_\Sigma(\tau))^* \boldsymbol{D}_\Sigma(\tau) = (\boldsymbol{D}_\Sigma)^2 = D_\tau^* D_\tau + \frac{1}{4} R_{h(\tau)} = -\Delta_{h(\tau)} + \frac{1}{4} R_{h(\tau)} \tag{36}$$

where $R_{h(\tau)}$ is the scalar curvature of $(\Sigma, h(\tau))$. The following theorem concerns the Cauchy problem on Block I for the linear equation (32) (see [33]) :

**Theorem 3.1**
*For any initial data* $\Psi_0 \in H_0^k(\Sigma;\mathbb{S}_D)$*,* $k \in \mathbb{N}$*, the system (32) has a unique solution* $\Psi$ *satisfying*

$$\Psi \in \bigcap_{l=0}^{k} C^l(\mathbb{R}_\tau; H_0^{k-l}(\Sigma;\mathbb{S}_D)). \tag{37}$$

*Moreover, there exits a propagator* $\mathcal{U}_D$ *such that:*
*-* $\mathcal{U}_D(\tau,\tau_0) : \Psi_0 \longmapsto \Psi(\tau)$*.*
*-* $\forall t, s\mathbb{R}$*,* $\mathcal{U}_D(t,s) \in \mathcal{L}(H_0^k(\Sigma;\mathbb{S}_D))$*,* $\mathcal{U}_D$ *is strongly continuous on* $\mathbb{R}_{ts}^2$ *to* $\mathcal{L}(H_0^k(\Sigma;\mathbb{S}_D))$*.*
*-* $\mathcal{U}_D(t,t) = Id$*,* $\mathcal{U}_D(t,s) = \mathcal{U}_D(t,r)\mathcal{U}_D(r,s)$ *for all* $t, s, r \in \mathbb{R}$*.*
*- we have in the sense of distributions on* $\mathbb{R} \times \Sigma$ *:*

$$\frac{\partial}{\partial \tau} \mathcal{U}_D(\tau,\tau_0)\Psi_0 = \mathcal{A}_D(\tau)\mathcal{U}_D(\tau,\tau_0)\Psi_0, \tag{38}$$

$$\frac{\partial}{\partial \tau_0} \mathcal{U}_D(\tau,\tau_0)\Psi_0 = -\mathcal{U}_D(\tau,\tau_0)\mathcal{A}_D(\tau_0)\Psi_0, \tag{39}$$

*- We have also the unitary evolution in* $L^2(\Sigma_\tau;\mathbb{S}_D)$ *:*

$$\|\Psi(\tau)\|_{L^2(\Sigma;\mathbb{S}_D)} = \|\Psi_0\|_{L^2(\Sigma;\mathbb{S}_D)} \tag{40}$$

*- There exists a continuous, strictly positive function* $\alpha_k$ *such that*

$$\|\Psi(\tau)\|_{k,\tau} \leq \alpha_k(\tau,\tau_0)\|\Psi_0\|_{k,\tau_0}, \quad \alpha_k(\tau,\tau) = 1, \tag{41}$$

*and a continuous, strictly positive function* $\kappa$ *such that*

$$\|\Psi(\tau)\|_{(k)} \leq \kappa_k(\tau,\tau_0)\|\Psi_0\|_{(k)}, \quad \kappa_k(\tau,\tau) = 1, \tag{42}$$

According to the $3+1$ decomposition, we consider the non linear problem (3) in the new equivalent form:

$$\frac{\partial \Psi}{\partial \tau} = \mathcal{A}_D(\tau)\Psi + \mathcal{J}_D(\Psi), \quad \mathcal{J}_D(\Psi) := i\frac{N}{\sqrt{2}} k(\Psi^* \gamma^0 \Psi)\gamma^0 \Psi. \tag{43}$$

In the sequel, we present a proof of the local Cauchy problem for the previous non linear equation with $H_0^2(\Sigma;\mathbb{S}_D)$ initial data. First, we prove the following lemma

**Lemma 3.1**
*There exists a constant* $C \geq 0$ *such that,*

$$\|\mathcal{J}_D(\Psi)\|_{(2)} \leq C\|\Psi\|_{(2)}^3, \quad \Psi, \Phi \in H_0^2(\Sigma;\mathbb{S}_D) \tag{44}$$

$$\|\mathcal{J}_D(\Psi) - \mathcal{J}_D(\Phi)\|_{(2)} \leq C\left(\|\Psi\|_{(2)}^2 + \|\Phi\|_{(2)}^2\right)\|\Psi - \Phi\|_{(2)} \tag{45}$$

**Proof:**
We only prove (45), since (44) is (45) with $\Phi \equiv 0$. We have

$$\begin{aligned}
&\mathcal{J}_D(\Psi) - \mathcal{J}_D(\Phi) \qquad (46)\\
&= \frac{N}{\sqrt{2}} k \left[ (\Psi^*\gamma^0\Psi)\gamma^0\Psi - (\Phi^*\gamma^0\Psi)\gamma^0\Psi + (\Phi^*\gamma^0\Psi)\gamma^0\Psi + (\Phi^*\gamma^0\Phi)\gamma^0\Psi - (\Phi^*\gamma^0\Phi)\gamma^0\Psi - (\Phi^*\gamma^0\Phi)\gamma^0\Phi \right],\\
&= \frac{N}{\sqrt{2}} k \left[ ([\Psi^* - \Psi^*]\gamma^0\Psi)\gamma^0\Psi + (\Phi^*\gamma^0[\Psi - \Phi])\gamma^0\Psi + (\Phi^*\gamma^0\Phi)\gamma^0[\Psi - \Phi] \right].
\end{aligned}$$

Now, we remark that $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$. Then, since $h(\tau)$ is equivalent to the Euclidian metric on $\mathbb{R}^3 \setminus \bar{B}(0,1)$, we have for $\Psi \in H_0^2(\Sigma; \mathbb{S}_D)$

$$\begin{aligned}
\|\Psi\|_{L^\infty(\Sigma;\mathbb{S}_D)} &= \|\tilde{\Psi}\|_{L^\infty(\mathbb{R}^+ \times S_\omega^2;\mathbb{S}_D)}\\
&= \|\tilde{\Psi}\|_{L^\infty(\mathbb{R}^3;\mathbb{S}_D)}\\
&\leq C_1 \|\tilde{\Psi}\|_{H^2(\mathbb{R}^3;\mathbb{S}_D)}\\
&= C_1 \|\tilde{\Psi}\|_{H^2(\mathbb{R}^+ \times S_\omega^2;\mathbb{S}_D)}\\
&= C_2 \|\Psi\|_{(2)}, \qquad C_1, C_2 > 0,
\end{aligned} \qquad (47)$$

where is $\tilde{\Psi}$ is the extension by zero on $\bar{B}(0,1)$ such that $\tilde{\Psi}_{|\Sigma} = \Psi$. Therefore, according to (46), (47) and since $N, k \in L^\infty$, we have:

$$\begin{aligned}
&\|\mathcal{J}_D(\Psi) - \mathcal{J}_D(\Phi)\|_{(2)}\\
&\leq C_3 \left( \|\Psi\|^2_{L^\infty(\Sigma;\mathbb{S}_D)} + \|\Psi\|_{L^\infty(\Sigma;\mathbb{S}_D)} \|\Phi\|_{L^\infty(\Sigma;\mathbb{S}_D)} + \|\Phi\|^2_{L^\infty(\Sigma;\mathbb{S}_D)} \right) \|\Psi - \Phi\|_{(2)}\\
&\leq C \left( \|\Psi\|^2_{(2)} + \|\Phi\|^2_{(2)} \right) \|\Psi - \Phi\|_{(2)}, \qquad C_3, C > 0.
\end{aligned} \qquad (48)$$

■

Now, we study the following problem

$$\left\{ \begin{array}{l} \Psi(\tau) = S(\Psi)(\tau),\\ S(\Psi)(\tau) := \mathcal{U}_D(\tau, \tau_0)\Psi_0 + \displaystyle\int_{\tau_0}^{\tau} \mathcal{U}_D(\tau, s)\mathcal{J}_D(\Psi(s))ds, \qquad \Psi \in C^0([\tau_0, \tau_0 + T[_\tau, H_0^2(\Sigma; \mathbb{S}_D)). \end{array} \right. \qquad (49)$$

to solve the local Cauchy problem

$$\left\{ \begin{array}{l} \frac{\partial \Psi}{\partial \tau} = \mathcal{A}_D(\tau)\Psi + \mathcal{J}_D(\Psi),\\ \Psi(\tau_0) = \Psi_0 \in H_0^2(\Sigma; \mathbb{S}_D), \qquad \Psi \in C^0([\tau_0, \tau_0 + T[_\tau, H_0^2(\Sigma; \mathbb{S}_D)). \end{array} \right. \qquad (50)$$

**Theorem 3.2**
*For $\Psi_0 \in H_0^2(\Sigma; \mathbb{S}_D)$, there exists $T > 0$ such that (43) admits a unique solution $\Psi$ such that*

$$\Psi(\tau_0) = \Psi_0 \in H_0^2(\Sigma; \mathbb{S}_D), \qquad \Psi \in C^0([\tau_0, \tau_0 + T[_\tau, H_0^2(\Sigma; \mathbb{S}_D)). \qquad (51)$$

**Proof :**
The operator $S$ in (49) is well defined if $\Psi_0 \in H_0^2(\Sigma; \mathbb{S}_D)$. Indeed, if $\Psi_0 \in H_0^2(\Sigma; \mathbb{S}_D)$ then $\mathcal{U}_D(\tau, \tau_0)\Psi_0 \in H_0^2(\Sigma; \mathbb{S}_D)$ by (42) and

$$\begin{aligned}
&\|\mathcal{U}_D(\tau, s+h)\mathcal{J}_D(\Psi(s+h)) - \mathcal{U}_D(\tau, s)\mathcal{J}_D(\Psi(s))\|_{(2)}, \qquad K := \max\{\kappa_2(\sigma, \tau),\ \sigma, \tau \in [\tau_0, \tau_0 + T]\},\\
&\leq \|\mathcal{U}_D(\tau, s+h)(\mathcal{J}_D(\Psi(s+h)) - \mathcal{J}_D(\Psi(s)))\|_{(2)} + \|(\mathcal{U}_D(\tau, s+h) - \mathcal{U}_D(\tau, s))\mathcal{J}_D(\Psi(s)))\|_{(2)}\\
&\leq K \|\mathcal{J}_D(\Psi(s+h)) - \mathcal{J}_D(\Psi(s))\|_{(2)} + K \|(\mathcal{U}_D(s, s+h) - 1)\mathcal{J}_D(\Psi(s))\|_{(2)}.
\end{aligned}$$

Hence $s \mapsto \mathcal{U}_D(\tau, s)\mathcal{J}_D(\Psi(s))$ is continuous on $H_0^2(\Sigma; \mathbb{S}_D)$, thanks to (44), (45) and since $\mathcal{U}_D$ is strongly continuous on $H_0^2(\Sigma; \mathbb{S}_D)$.

Moreover, $C^0([\tau_0,\tau_0+T[_\tau, H_0^2(\Sigma;\mathbb{S}_D))$ is stable by $S$ :

$$\begin{aligned}\|S(\Psi)(\tau+h)-S(\Psi)(\tau)\|_{(2)} &\leq \|\mathcal{U}_D(\tau+h,\tau_0)\Psi_0-\mathcal{U}_D(\tau,\tau_0)\Psi_0\|_{(2)} \\ &+\int_{\tau_0}^{\tau}\|\mathcal{U}_D(\tau+h,s)\mathcal{J}_D(\Psi(s))-\mathcal{U}_D(\tau,s)\mathcal{J}_D(\Psi(s))\|_{(2)}ds \\ &+\int_{\tau+h}^{\tau}\|\mathcal{U}_D(\tau+h,s)\mathcal{J}_D(\Psi(s))\|_{(2)}ds\end{aligned} \tag{52}$$

This norm vanishes as $h\to 0$, since $\mathcal{U}$ is strongly continuous on $H_0^2(\Sigma;\mathbb{S}_D)$ for the first term on right on side, thanks to the same property, the Lebesgue theorem and (45) for the second. The last is bounded by $Kh\sup_{s\in[\tau,\tau+h]}\|\Psi(s)\|^3_{H_0^2(\Sigma;\mathbb{S}_D)}$, $K:=\max\{\kappa_2(\sigma,\tau),\ \sigma,\tau\in[\tau_0,\tau_0+T]\}$ thanks to (44).
Now, we define the convex closed of $C^0([\tau_0,\tau_0+T[_\tau, H_0^2(\Sigma;\mathbb{S}_D))$:

$$V_{T,\Psi_0}:=\left\{\Psi\in C^0([\tau_0,\tau_0+T[_\tau, H_0^2(\Sigma;\mathbb{S}_D));\Psi(\tau_0)=\Psi_0,\ \|\Psi\|_T\leq 2K\|\Psi_0\|_{(2)}\right\}, \tag{53}$$

$$K:=\max\{\kappa_2(\sigma,\tau),\ \sigma,\tau\in[s,s+T]\} \tag{54}$$

with

$$\|\Psi\|_T=\sup_{[\tau_0,\tau_0+T[_\tau}\|\Psi\|_{(2)}. \tag{55}$$

For $T$ small $S(V_{T,\Psi_0})\subset V_{T,\Psi_0}$. Indeed, according to (45) and (53), we have

$$\|S(\Psi)\|_T\leq K(1+8TCK^2\|\Psi_0\|^2_{(2)})\|\Psi_0\|_{(2)},\quad \Psi\in V_{T,\Psi_0} \tag{56}$$

and we choose

$$T<(8CK^2\|\Psi_0\|^2_{(2)})^{-1}. \tag{57}$$

Moreover, we obtain with (45):

$$\|S(\Psi)-S(\Phi)\|_T\leq TK\|\mathcal{J}_D(\Psi)-\mathcal{J}_D(\Phi)\|_T\leq 4TK^2C\|\Psi_0\|^2_{(2)}\|\Psi-\Phi\|_T,\quad \Psi,\Phi\in V_{T,\Psi_0} \tag{58}$$

Then, if we choose

$$T<min(8CK^2\|\Psi_0\|^2_{(2)})^{-1},(4TK^2C\|\Psi_0\|^2_{(2)})^{-1}) \tag{59}$$

by the Banach fixed point theorem there exists a solution of (49).

Now we study the uniqueness of this problem. Given $T>0$, $\Psi_0\in H_0^2(\Sigma;\mathbb{S}_D)$ and two solutions $\Psi_1,\Psi_2\in C^0([\tau_0,\tau_0+T[_\tau, H_0^2(\Sigma;\mathbb{S}_D))$ associated to $\Psi_0$, then with $K:=\max\{\kappa_2(\sigma,\tau),\ \sigma,\tau\in[\tau_0,\tau_0+T]\}$

$$\begin{aligned}\|\Psi_1(\tau)-\Psi_2(\tau)\|_{(2)} &\leq K\int_{\tau_0}^{\tau}\|\mathcal{J}_D(\Psi_1(s))-\mathcal{J}_D(\Psi_2(s))\|_{(2)}ds, \\ &\leq K\left(\|\Psi\|_T^2+\|\Phi\|_T^2\right)\int_{\tau_0}^{\tau}\|\Psi_1(s)-\Psi_2(s)\|_{(2)}ds.\end{aligned} \tag{60}$$

Then by Gronwall lemma we have $\Psi_1=\Psi_2$.

First, we prove that the problems (49) and (50) are equivalent. If $\Psi$ is solution of (49) then $\Psi$, satisfies equation (50). Indeed, according to the properties of $\mathcal{U}_D$ we have

$$\begin{aligned}\frac{\partial\Psi}{\partial\tau}(\tau) &= \mathcal{A}_D(\tau)\mathcal{U}_D(\tau,\tau_0)\Psi+\int_{\tau_0}^{\tau}\mathcal{A}_D(\tau)\mathcal{U}_D(\tau,s)\mathcal{J}_D(\Psi(s))ds+\mathcal{J}_D(\Psi(\tau)) \\ &= \mathcal{A}_D(\tau)\Psi(\tau)+\mathcal{J}_D(\Psi(\tau))\end{aligned}$$

since $s\mapsto\mathcal{A}_D(\tau)\mathcal{U}_D(\tau,s)\mathcal{J}_D(\Psi(s))$ is bounded on $H_0^2(\Sigma;\mathbb{S}_D)$ by (44), (42) and also $\mathcal{A}_D(\tau)$ is closed for each $\tau$. The inverse is straightforward. The uniqueness follows from the Gronwall lemma. ■

**Remark 3.1**
*1) Theorem 3.2 is in fact valid for $\Psi_0 \in H_0^s(\Sigma;\mathbb{S}_D)$, $s \geq 2$. Indeed, it is easy to prove thanks to the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ an equivalent lemma of Lemma 3.1 for $H_0^s(\Sigma;\mathbb{S}_D)$. The proof of the theorem with $\Psi_0 \in H_0^s(\Sigma;\mathbb{S}_D)$ is essentially the same.*
*2) Theorem 3.2 is still valid with the following non linearity :*

$$\mathcal{J}_D^p(\Psi) := i\frac{N}{\sqrt{2}}k|(\Psi^*\gamma^0\Psi)|^{\frac{p-1}{2}}\gamma^0\Psi, \quad p \geq 3. \tag{61}$$

*Then, we obtain the following estimates :*

$$\|\mathcal{J}_D^p(\Psi)\|_{(s)} \leq C\|\Psi\|_{(s)}^p, \qquad \Psi,\Phi \in H_0^s(\Sigma;\mathbb{S}_D), \tag{62}$$

$$\|\mathcal{J}_D^p(\Psi) - \mathcal{J}_D^p(\Phi)\|_{(s)} \leq C\left(\|\Psi\|_{(s)}^{p-1} + \|\Phi\|_{(s)}^{p-1}\right)\|\Psi - \Phi\|_{(s)}. \tag{63}$$

*As above, the proof with this non linearity remains essentially the same.*

# 4 Local Cauchy problem for the Dirac-Klein-Gordon equation outside a Kerr Black hole

In this section we study the local Cauchy problem for the Dirac-Klein-Gordon equation (4). As above, we use the $3+1$ decomposition to defined an evolution problem in $\mathcal{B}_I$. To prove the existence of local solutions, we use a Duhamel formula and a Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, $s \geq 2$.

According to the $3+1$ decomposition, the previous section and the following definition

$$\Box_g = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) \tag{64}$$

equation (4) is equivalent to

$$\frac{\partial\Psi}{\partial\tau} = \mathcal{A}_D(\tau)\Psi + i\frac{N}{\sqrt{2}}\Phi\mathcal{V}_2\Psi. \tag{65}$$

$$\frac{\partial^2\Phi}{\partial\tau^2} = \mathcal{A}_{KG}(\tau)\Phi + N^2\Psi^*\mathcal{V}_3\Psi, \;\; \mathcal{A}_{KG}(\tau) := N^2\Delta_h - N^2 m_{kg}^2 \tag{66}$$

where

$$\Delta_h := \frac{1}{N\sqrt{|h|}}\frac{\partial}{\partial x^a}\left(N\sqrt{|h|}h^{ab}\frac{\partial}{\partial x^b}\right) \tag{67}$$

and $\mathcal{A}_D(\tau)$ defined in (32). Now we put this equation in hamiltonian form such that

$$\frac{\partial U}{\partial\tau} = \mathcal{A}_{DKG}(\tau)U + \mathcal{J}_{DKG}(U), \quad U =^t (\Psi,\Phi,\partial_\tau\Phi). \tag{68}$$

where

$$\mathcal{A}_{DKG}(\tau) := \begin{pmatrix} \mathcal{A}_D(\tau) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \mathcal{A}_{KG}(\tau) & 0 \end{pmatrix}, \quad \mathcal{J}_{DKG}(U) := \begin{pmatrix} i\frac{N}{\sqrt{2}}\Phi\mathcal{V}_2\Psi \\ 0 \\ N^2\Psi^*\mathcal{V}_3\Psi \end{pmatrix}. \tag{69}$$

First, we remark some properties for the linear part of the equation (66) ($i.e$ $\mathcal{V}_3 = 0$). According to a result due to Leray [25], we have for this linear equation :

**Theorem 4.1**
*For initial data* $\Phi_0, \Phi_1 \in C_0^\infty(\Sigma)$ *in* $\tau_0 \in \mathbb{R}$*, the equation (66) with* $V_3 = 0$ *has a solution* $\Phi \in C^\infty(\mathbb{R}_\tau, C_0^\infty(\Sigma))$ *satisfying* $\Phi(\tau_0) = \Phi_0$ *and* $\partial_\tau \Phi(\tau_0) = \Phi_1$.

The proof of the following proposition consists in multiplying the equation by $\partial_\tau \Phi$ and integrating by part on $]\tau_0, \tau[\times\Sigma$:

**Proposition 4.1**
*There exists a continuous, strictly positive function* $\xi$ *such that* $\xi(\tau,\tau) = 1$ *and for each* $\Phi \in C^\infty(\mathbb{R}_\tau, C_0^\infty(\Sigma))$ *solution of the equation (66) with* $V_3 = 0$*, we have for any* $\tau, \tau_0 \in \mathbb{R}$

$$E_{3+1}(\Phi, \tau) \leq \xi(\tau, \tau_0) E_{3+1}(\Phi, \tau_0) \tag{70}$$

*with*

$$E_{3+1}(\Phi, \tau) := \int_{\Sigma_\tau} \left( |\partial_t \Phi|^2 + N^2 h^{ab} \partial_a \Phi \partial_b \bar{\Phi} + N^2 m_{kg}^2 |\Phi|^2 \right) \frac{1}{N} dVol. \tag{71}$$

According to Proposition 2.1, we consider $\eta = h(0)$ on $\bar{\Sigma}$ and we introduce the Hilbert space $\mathcal{H}_0^{1,0}(\Sigma)$ as the completion of $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$ for the norm

$$\|(u,v)\|^2_{\lfloor 1,0 \rfloor} := \int_\Sigma \left( |v|^2 - \mathcal{A}_{KG} u \bar{u} \right) \frac{1}{N} dVol = \int_\Sigma \left( |v|^2 + N^2 |\nabla u|^2 + N^2 m_{kg}^2 |u|^2 \right) \frac{1}{N} dVol, \tag{72}$$
$$|\nabla u|^2 = \eta^{ab} \partial_a u \partial_b \bar{u}.$$

With these two last results, we deduce the

**Proposition 4.2**
*For any initial data* $U_0 := {}^t(\Phi_0, \Phi_1) \in \mathcal{H}_0^{1,0}(\Sigma)$ *in* $\tau_0 \in \mathbb{R}$*, the equation (66) with* $V_3 = 0$ *has an unique solution* $U \in C^0(\mathbb{R}_\tau, \mathcal{H}_0^{1,0}(\Sigma))$ *satisfying* $U(\tau_0) = {}^t(\Phi_0, \Phi_1)$*. Moreover, we have the existence of a propagator* $\mathcal{U}_{KG}$ *such that :*
*- For all* $\tau, \sigma \in \mathbb{R}$*,* $\mathcal{U}_{KG}(\tau, \sigma) \in \mathcal{L}(\mathcal{H}_0^{1,0})$*,* $\|\mathcal{U}_{KG}(\tau,\sigma)\|_{\mathcal{L}(\mathcal{H}_0^{1,0}(\Sigma))} \leq \xi(\sigma, \tau)$ *where* $\xi$ *is the function defined in (70).*
*-* $\mathcal{U}_{KG}(\tau,\tau) = 1$*,* $\mathcal{U}_{KG}(\tau, s) = \mathcal{U}_{KG}(t,r)\mathcal{U}_{KG}(r,s)$ *for all* $t, s, r \in \mathbb{R}$*.*
*- We have in the sense of distributions on* $\mathbb{R} \times \Sigma$ *:*

$$\frac{\partial}{\partial \tau} \mathcal{U}_{KG}(\tau, \tau_0) U_0 = \mathcal{M}_{\mathcal{A}_{KG}}(\tau) \mathcal{U}_{KG}(\tau, \tau_0) U_0, \qquad \mathcal{M}_{\mathcal{A}_{KG}} = \begin{pmatrix} 0 & 1 \\ \mathcal{A}_{KG} & 0 \end{pmatrix}, \tag{73}$$
$$\frac{\partial}{\partial \tau_0} \mathcal{U}_{KG}(\tau, \tau_0) U_0 = -\mathcal{U}_{KG}(\tau, \tau_0) \mathcal{M}_{\mathcal{A}_{KG}}(\tau_0) U_0. \tag{74}$$

We introduce $\tilde{\mathcal{A}}_{KG}$ such that

$$\tag{75}$$
$$A_{KG} := N^2 \Delta_\eta - \mathbf{1}. \tag{76}$$

Now, we define $\mathcal{H}_0^{2,1}(\Sigma)$ as the completion of $C_0^\infty(\Sigma) \oplus C_0^\infty(\Sigma)$ for the norm

$$\begin{aligned}
\|(u,v)\|^2_{\lfloor 2,1 \rfloor} &:= \int_\Sigma \left( |\tilde{\mathcal{A}}_{KG} u|^2 - \mathcal{A}_{KG} u \bar{u} - \mathcal{A}_{KG} v \bar{v} \right) \frac{1}{N} dVol, \\
&= \int_\Sigma \left( N^2 |\nabla v|^2 + N^2 m_{kg}^2 |v|^2 + |N^2 \Delta_\eta u - u|^2 + N^2 |\nabla u|^2 + N^2 m_{kg}^2 |u|^2 \right) \frac{1}{N} dVol \\
&= \int_\Sigma \left( N^2 |\nabla v|^2 + N^2 m_{kg}^2 |v|^2 + N^4 |\Delta_\eta u|^2 + (1 + m_{kg}^2) N^2 |\nabla u|^2 + (m_{kg}^2 N^2 + 1)|u|^2 \right) \frac{1}{N} dVol.
\end{aligned}$$

But, for a regular solution $U^{KG}$ of (66) with $V_3=0$ associated to the initial data $U_0^{KG}\in C_0^\infty(\Sigma)\oplus C_0^\infty(\Sigma)$, we have

$$\begin{aligned}\partial_\tau(\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau)U^{KG}(\tau)) &= \mathcal{M}_{\mathcal{A}_{KG}}(\tau)(\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau)U^{KG}(\tau))\\ &+\left((\partial_\tau\mathcal{M}_{\mathcal{A}_{KG}}(\tau))+[\mathcal{M}_{\mathcal{A}_{KG}}(\tau),\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau)]\right)U^{KG}(\tau)\end{aligned}$$

where

$$[\mathcal{M}_{\mathcal{A}_{KG}}(\tau),\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau)]=\begin{pmatrix}\tilde{\mathcal{A}}_{KG}-\mathcal{A}_{KG} & 0\\ 0 & \mathcal{A}_{KG}-\tilde{\mathcal{A}}_{KG}\end{pmatrix},\quad \mathcal{D}:=(N^2-1)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{77}$$

Hence, we obtain the integral formula

$$\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau)U^{KG}(\tau)=\mathcal{U}_{KG}(\tau,\tau_0)(\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau_0)U^{KG}(\tau_0))+\int_{\tau_0}^{\tau}\mathcal{U}_{KG}(\tau,\sigma)G(\sigma)d\sigma, \tag{78}$$

with

$$G(\sigma):=\left((\partial_\tau\mathcal{M}_{\mathcal{A}_{KG}}(\sigma))+[\mathcal{M}_{\mathcal{A}_{KG}}(\tau),\mathcal{M}_{\tilde{\mathcal{A}}_{KG}}(\tau)]\right)U^{KG}(\sigma). \tag{79}$$

But thanks to Proposition 2.1 we have

$$\|G(\sigma)\|_{\lfloor 1,0\rfloor}\le C(\sigma)\|U^{KG}(\sigma)\|_{\lfloor 2,1\rfloor} \tag{80}$$

where $C$ is a continuous positive function on $\mathbb{R}$ independent of $U^{KG}$. Hence, by formula (78) and the Gronwall lemma we obtain the estimate

$$\|U^{KG}(\tau)\|_{\lfloor 2,1\rfloor}\le C'(\tau,\tau_0)\|U_0^{KG}\|_{\lfloor 2,1\rfloor}, \tag{81}$$

where $C'$ is a continuous function such that $C'(\tau,\tau)=1$. Now, we introduce $H_0^{2,1}(\Sigma)$ as the completion of $C_0^\infty(\Sigma)\oplus C_0^\infty(\Sigma)$ for the norm

$$\|(u,v)\|_{(2,1)}^2:=\int_\Sigma\left(N^2|\nabla v|^2+|v|^2+N^4|\Delta_\eta u|^2+N^2|\nabla u|^2+|u|^2\right)\frac{1}{N}dVol. \tag{82}$$

This space, smaller than $\mathcal{H}_0^{2,1}(\Sigma)$ allow us to use the Sobolev embedding. Clearly $H_0^{2,1}(\Sigma)\hookrightarrow\mathcal{H}_0^{2,1}(\Sigma)$. Moreover, if we consider the initial data $U_0^{KG}\in C_0^\infty(\Sigma)\oplus C_0^\infty(\Sigma)$ in some initial time $\tau_0\in\mathbb{R}$, we have $U^{KG}\in C^\infty(\mathbb{R}_\tau;C_0^\infty(\Sigma)\oplus C_0^\infty(\Sigma))$ and

$$\|U^{KG}(\tau)\|_{(2,1)}\le C_1\left(\|U^{KG}(\tau)\|_{\lfloor 2,1\rfloor}+\|\Phi(\tau)\|_{L^2(\Sigma;N^{-1}dVol)}+\|\partial_\tau\Phi(\tau)\|_{L^2(\Sigma;N^{-1}dVol)}\right),\quad C_1>0. \tag{83}$$

But

$$\begin{aligned}\|\Phi(\tau)\|_{L^2(\Sigma;N^{-1}dVol)} &\le \|\Phi(\tau_0)\|_{L^2(\Sigma;N^{-1}dVol)}+\int_{\tau_0}^{\tau}\|\partial_\tau\Phi(\sigma)\|_{L^2(\Sigma;N^{-1}dVol)}d\sigma &(84)\\ &\le \|U^{KG}(\tau_0)\|_{(2,1)}+\int_{\tau_0}^{\tau}\|U^{KG}(\sigma)\|_{(2,1)}d\sigma &(85)\end{aligned}$$

and

$$\begin{aligned}\|\partial_\tau\Phi(\tau)\|_{L^2(\Sigma;N^{-1}dVol)} &\le \|\partial_\tau\Phi(\tau_0)\|_{L^2(\Sigma;N^{-1}dVol)}+\int_{\tau_0}^{\tau}\|\partial_\tau^2\Phi(\sigma)\|_{L^2(\Sigma;N^{-1}dVol)}d\sigma &(86)\\ &\le \|U^{KG}(\tau_0)\|_{(2,1)}+\int_{\tau_0}^{\tau}\|\mathcal{A}_{KG}\Phi(\sigma)\|_{L^2(\Sigma;N^{-1}dVol)}d\sigma &(87)\\ &\le \|U^{KG}(\tau_0)\|_{(2,1)}+\int_{\tau_0}^{\tau}\|U^{KG}(\sigma)\|_{(2,1)}d\sigma. &(88)\end{aligned}$$

Hence, thanks to (81) and the two previous estimates, we deduce with (83) that

$$\|U^{KG}(\tau)\|_{(2,1)} \leq C'(\tau,\tau_0)\|U^{KG}(\tau_0)\|_{(2,1)} + C_2\|U^{KG}(\tau_0)\|_{(2,1)} + C_2\int_{\tau_0}^{\tau}\|U^{KG}(\sigma)\|_{(2,1)}d\sigma, \quad C_2>0 \tag{89}$$

and by the Gronwall lemma

$$\|U^{KG}(\tau)\|_{(2,1)} \leq C''(\tau,\tau_0)\|U_0^{KG}\|_{(2,1)}, \tag{90}$$

with $C''$ a continuous positive function on $\mathbb{R}$ independent of $U^{KG}$. This last estimate allows us to extend $\mathcal{U}_{KG}$ as a propagator on $H_0^{2,1}(\Sigma)$. Hence, with the Theorem 3.1 and the preview result we deduce the existence of a progator for the linear part of equation (68) on $H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma)$:

**Proposition 4.3**
*For any initial data $U_0\in H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma)$, the system (68) with $\mathcal{J}_{DKG}=0$ has a unique solution $U$ satisfying*

$$U\in C^0(\mathbb{R}_\tau; H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma)). \tag{91}$$

*Moreover, there exits a propagator $\mathcal{U}_{DKG}$ such that:*
*- $\mathcal{U}_{DKG}(\tau,\tau_0): U_0 \longmapsto U(\tau), \quad \mathcal{U}_{DKG} := {}^t(\mathcal{U}_D,\mathcal{U}_{KG})$.*
*- $\forall t,s\in\mathbb{R}$, $\mathcal{U}_{DKG}(t,s)\in\mathcal{L}(H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma))$*
*- $\mathcal{U}_{DKG}$ is strongly continuous on $\mathbb{R}^2_{ts}$ to $\mathcal{L}(H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma))$.*
*- $\mathcal{U}_{DKG}(t,t)=Id$, $\mathcal{U}_{DKG}(t,s)=\mathcal{U}_{DKG}(t,r)\mathcal{U}_{DKG}(r,s)$ for all $t,s,r\in\mathbb{R}$.*
*- There exists a continuous, strictly positive function $\kappa_{DKG}$ such that*

$$\|U(\tau)\|_{(2,2,1)} := \|\Psi(\tau)\|_{(2)} + \|(\Phi(\tau),\partial_\tau\Phi(\tau))\|_{(2,1)} \leq \kappa_{DHG}(\tau,\tau_0)\|U_0\|_{(2,2,1)}, \qquad \kappa_{DKG}(\tau,\tau)=1. \tag{92}$$

To study the non linear Cauchy problem for (DKG), we establish the following lemma about the continuity of $\mathcal{J}_{DKG}$ on $H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma)$ :

**Lemma 4.1**
*There exists a constant $C\geq 0$ such that, $U_i=(\Psi_i,V_i)=(\Psi_i,\Phi_i,\partial_\tau\Phi_i)\in H_0^2(\Sigma;\mathbb{S}_D)\oplus H_0^{2,1}(\Sigma)$ and $\cup_i supp(\Phi_i)\subset [R,+\infty[\times S_\omega$, $R>0$*

$$\|\mathcal{J}_{DKG}(U_1)\|_{(2,2,1)} \leq C\|U_1\|^2_{(2,2,1)}, \tag{93}$$

$$\|\mathcal{J}_{DKG}(U_1)-\mathcal{J}_{DKG}(U_2)\|_{(2,2,1)} \leq C\left(\|U_1\|_{(2,2,1)}+\|U_2\|_{(2,2,1)}\right)\|U_1-U_2\|_{(2,2,1)} \tag{94}$$

**Proof:**
We prove (94), indeed (93) is (94) with $U_2\equiv 0$. We have

$$\mathcal{J}_{DKG}(U_1)-\mathcal{J}_{DKG}(U_2) = \begin{pmatrix} i\frac{N}{\sqrt{2}}\left(\Phi_2\mathcal{V}_2(\Psi_1-\Psi_2)+(\Phi_1-\Phi_2)\mathcal{V}_2\Psi_1\right) \\ 0 \\ N^2\left(\Psi_2^*\mathcal{V}_3(\Psi_1-\Psi_2)+(\Psi_1^*-\Psi_2^*)\mathcal{V}_3\Psi_1\right)\end{pmatrix}. \tag{95}$$

We estimate the first component of the previous difference. We introduce $H_0^2(\Sigma)$ as the completion of $C_0^\infty(\Sigma)$ in the norm

$$\|\Phi\|_{H^2(\Sigma)} := \int_\Sigma |\Phi|^2+|\nabla\Phi|^2+|\underline{\Delta}_\eta\Phi|^2 dVol, \qquad \underline{\Delta}_\eta := \frac{1}{\sqrt{|\eta|}}\frac{\partial}{\partial x^a}\left(\sqrt{|\eta|}\eta^{ab}\frac{\partial}{\partial x^b}\right). \tag{96}$$

Since $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ and $h(\tau)$ is equivalent to the Euclidian metric on $\mathbb{R}^3\setminus\bar{B}(0,1)$, we obtain as in Lemma 3.1

$$\|\Phi\|_{L^\infty(\Sigma)} \leq C_1\|\Phi\|_{H^2(\Sigma)}, \; C_1>0, \quad \Phi\in H_0^2(\Sigma). \tag{97}$$

Moreover, for $\Phi \in C_0^\infty(\Sigma)$

$$N\Delta_\eta \Phi = N\underline{\Delta}_\eta \Phi + \nabla N.\nabla\Phi \tag{98}$$

and

$$\underline{\Delta}_\eta (N\Phi) = N\underline{\Delta}_\eta \Phi + 2\nabla N.\nabla\Phi + (\underline{\Delta}_\eta N)\Phi. \tag{99}$$

Therefore, for $V = {}^t(\Phi, \partial_\tau\Phi) \in H_0^{2,1}(\Sigma)$ with $supp(\Phi) \subset [R, +\infty[\times S_\omega$, $R > 0$ and since on $supp(\Phi)$ there exists $C_i > 0$ such that $C_1 \leq N \leq C_2$ we have

$$\|N\Phi\|_{L^\infty(\Sigma)} \leq C_2\|V\|_{(2,1)},\ C_2 > 0,\quad V = {}^t(\Phi, \partial_\tau\Phi) \in H_0^{2,1}(\Sigma). \tag{100}$$

Hence

$$\begin{aligned}\left\| i\frac{N}{\sqrt{2}}\left(\Phi_2\mathcal{V}_2(\Psi_1-\Psi_2)+(\Phi_1-\Phi_2)\mathcal{V}_2\Psi_1\right)\right\|_{(2)} &\leq C_2(\|V_2\|_{(2,1)}\|\Psi_1-\Psi_2\|_{(2)} + \|V_1-V_2\|_{(2,1)}\|\Psi_1\|_{(2)}) \\ &\leq C_2\left(\|U_1\|_{(2,2,1)} + \|U_2\|_{(2,2,1)}\right)\|U_1-U_2\|_{(2,2,1)}. \end{aligned}\tag{101}$$

Moreover with (47), we have

$$\begin{aligned}&\|(0, N^2\left(\Psi_2^*\mathcal{V}_3(\Psi_1-\Psi_2)+(\Psi_1^*-\Psi_2^*)\mathcal{V}_3\Psi_1\right))\|_{(2,1)} \\ &\qquad\leq C_3\left(\|U_1\|_{(2,2,1)} + \|U_2\|_{(2,2,1)}\right)\|U_1-U_2\|_{(2,2,1)},\qquad C_3 > 0\end{aligned}\tag{102}$$

and with (101) we deduce the result. ■

According to Theorem 4.3, $\mathcal{U}_{DKG} \in \mathcal{L}(H_0^2(\Sigma;\mathbb{S}_D) \oplus H_0^{2,1}(\Sigma))$ and satisfies (92). Moreover, for $U_0 := (\Psi_0, \Phi_0, \Phi_1) \in H_0^2(\Sigma;\mathbb{S}_D) \oplus H_0^{2,1}(\Sigma)$ at $\tau_0 > 0$ and $\cup_i supp(\Phi_i) \subset [R, +\infty[\times S_\omega$, $R > 0$, we have $supp(\mathcal{U}(\tau,\tau_0)U_0) \subset [R'(R,\tau,\tau_0), +\infty[\times S_\omega$, $R'(R,\tau,\tau_0) < R$ since the system (68) with $\mathcal{J}_{DKG} = 0$ is hyperbolic. Hence, in the integral formulation of the Cauchy problem, we can apply lemma 4.1 for $\mathcal{U}(\tau,\tau_0)U_0$. By an identical proof of Theorem 3.2, we have the following theorem :

**Theorem 4.2**
*For $U_0 := (\Psi_0, \Phi_0, \Phi_1) \in H_0^2(\Sigma;\mathbb{S}_D) \oplus H_0^{2,1}(\Sigma)$ at $\tau_0 > 0$ and $\cup_i supp(\Phi_i) \subset [R, +\infty[\times S_\omega$, $R > 0$, there exists $T > 0$ such that system (68) admits a unique solution $U$ such that*

$$U(\tau_0) = U_0 \in H_0^2(\Sigma;\mathbb{S}_D) \oplus H_0^{2,1}(\Sigma),\qquad U \in C^0([\tau_0, \tau_0 + T[_\tau, H_0^2(\Sigma;\mathbb{S}_D) \oplus H_0^{2,1}(\Sigma)). \tag{103}$$

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