The Hawking effect for a collapsing star in an initial state of KMS type

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Abstract. - We prove the Hawking effect for a gravitational collapse of charged star, stationary in the past and collapsing to a black hole in the future. In the past, the ground state of the Dirac fields is given by a KMS state with unspecified temperature.

1 Introduction.

This article extends our previous investigation [14] about the Hawking effect [11] for the Dirac field. In [14] we considered a charged star stationnary in the past, and collapsing to a black hole in the framework of the semiclassical approximation where the back-reaction of the field on the metric is neglected. Furthermore, the ground state in the past was given by the Boulware vacuum. In this new work and always for the semiclassical regime, we study the same collapsing star, but in the past, we consider a ground state given by a KMS state with unspecified temperature. In the case of collapse in expending universe the temperature physically relevant is that of Gibbons-Hawking associated to the cosmological horizon [10]. As in [14], we prove the emergence of thermal state coming from the future black hole which is independent of the story of the collapse and the nature of the star surface. Moreover, with the results of this paper and the previous, we also remark that the choice of the ground state in the past does not modify the caracteristic of the flux of particles coming from the horizon of the future black hole.

During the collapse the star becomes a black hole. This black hole is described in term the Schwarzschild coordinates (t, r, ω) as the globally hyperbolic manifold $(\mathcal{M}_{\rm bh}, g)$, (see for example [12], [16], [20])

$$\begin{aligned}
&\mathcal{M}_{\rm bh} := \mathbb{R}_t \times]r_0, r_+[_r \times S^2_\omega, \quad 0 < r_0 < r_+ \leq +\infty, \\
&g_{\mu\nu} dx^\mu dx^\nu = F(r) dt^2 - F^{-1}(r) dr^2 - r^2 d\omega^2, \\
&d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2, \ \omega = (\theta, \varphi) \in [0, \pi] \times [0, 2\pi[, \\
&F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3},
\end{aligned} \tag{1}$$

where $Q \in \mathbb{R}$, $M > 0$ and $\Lambda \geq 0$ are respectively the electric charge, the mass, the cosmological constant. Here r_0 and r_+ are the radius of the horizon of the black-hole and the radius of the cosmological horizon and moreover

$$F(r_0) = F(r_+) = 0, \quad 2\kappa_0 = F'(r_0) > 0, \quad 2\kappa_+ = F'(r_+) < 0, \ r \in]r_0, r_+[\Rightarrow F(r) > 0, \tag{2}$$

with κ_0, κ_+ the surface gravity at the black hole horizon and at the cosmological horizon. If the cosmological constant $\Lambda = 0$, then $(\mathcal{M}_{\mathrm{bh}}, g)$ describes the asymptotically flat space time of Reissner-Nordstrøm with

$$F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad 0 < |Q| \leq M,$$
$$r_0 = M + \sqrt{M^2 - Q^2}, \quad r_+ = +\infty.$$

We introduce the Regge-Wheeler coordinate such that

$$\frac{dr_*(r)}{dr} = F^{-1}. \tag{3}$$

With this new radial coordinate, the horizons are pushed away at infinities:

$$r_*(r) \to -\infty \quad \Longleftrightarrow \quad r \to r_0, \quad \Lambda \geq 0$$
$$r_*(r) \to +\infty \quad \Longleftrightarrow \quad r \to r_+, \quad \Lambda > 0, \quad r_*(r) \to +\infty \quad \Longleftrightarrow \quad r \to +\infty, \quad \Lambda = 0.$$

Hence, we define the space time outside the collapsing star with mass $M > 0$ and r_*-radius $z(t),\ t \in \mathbb{R}$ in an expanding or asymptotically flat universe, such that :

$$\mathcal{M}_{\mathrm{coll}} := \left\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega, \quad r_* \geq z(t)\right\}. \tag{4}$$

The reasonable assumptions of generic collapsing examined in [1] leads to the following properties for $z(t)$:

$$z \in C^2(\mathbb{R});\ \forall t \in \mathbb{R}, \quad -1 < \dot{z}(t) \leq 0, \quad t \leq 0 \Rightarrow z(t) = z(0) < 0. \tag{5}$$
$$z(t) = -t - C_{\kappa_0} e^{-2\kappa_0 t} + \varpi(t),\ C_{\kappa_0} > 0,\ |\varpi(t)| + |\dot{\varpi}(t)| = \mathcal{O}\left(e^{-4\kappa_0 t}\right),\ t \to +\infty. \tag{6}$$

According to the Birkhoff theorem and since the spherical symmetry of the star is maintained during the collapse, the metric on $\mathcal{M}_{\mathrm{coll}}$ is just the Lorentzian metric g defined in (1).

On $(\mathcal{M}_{\mathrm{coll}}, g)$ we consider the Dirac equation for a fermion of mass $m > 0$ and charge $q \in \mathbb{R}$:

$$i\gamma^\mu \bar{\nabla}_\mu \Psi + iq\frac{Q}{r}\Psi - m\Psi = 0. \tag{7}$$

The term $\frac{Q}{r}$ is the electromagnetic potential since we take electromagnetic interactions between the field and the charged star into account. Here γ^μ are the Dirac matrices in curved space time and $\bar{\nabla}_\mu$ the spinor fields covariant derivative. Our model of the star is very simple and very convenient since our star is in fact a mirror. This assumptions enable us to avoid to treat the different interaction and behavior of the fluid inside the star during the collapse. Therefore, on the star surface,

$$\mathcal{S} := \left\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega, \quad r_* = z(t)\right\},$$

we put the following conservative boundary condition, written for $(t, r_*, \omega) \in \mathcal{S}$, as

$$n_j \gamma^j \Psi(t, r_*, \omega) = ie^{i\nu\gamma^5}\Psi(t, r_*, \omega), \quad \gamma^5 := -i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{8}$$

where n_j is the outgoing normal of subset of $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega$ and ν the chiral angle. We suppose for the technical reasons that $\nu \in \mathbb{R}$ if $r_+ < +\infty$, and $\nu \neq (2k+1)\pi$, $k \in \mathbb{Z}$ if $r_+ = +\infty$. This conservative boundary condition is the generalized *MIT bag* boundary condition [5] causes a reflexion of the fields on the star surface.

In the second part of this work, we state the theorem giving a solution of the mixed hyperbolic problem (7)-(8) with the help of a propagator. In this same part, we also introduce the useful wave operators outside the future black-hole. In the fifth part, we state and interpret the main theorem of this work using the Quantum Field Theory. To do this, we construct the local algebra of observable $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ as in [6] and use the wave operators of the second part. Finally in the last section, we expose the mathematical proof of the main theorem of this article.

2 Classical fields.

2.1 Dirac equation.

By using the definition (1) and a calculation from [2] and [17] for equation (7), we set in a hamiltonian form the mixte hyperbolic mixed problem on $(\mathcal{M}_{\text{coll}}, g)$ related to (7) and (8) :

$$\partial_t \Psi = i\boldsymbol{D}_t \Psi, \quad z(t) < r_*, \tag{9}$$

$$\frac{\dot{z}\gamma^0 - \gamma^1}{\sqrt{1-\dot{z}^2}} \Psi(t, z(t)) = ie^{i\nu\gamma^5}\Psi(t, z(t)) \tag{10}$$

$$\Psi(t = s, .) = \Psi_s(.) \in \boldsymbol{L}^2_s, \tag{11}$$

where $\boldsymbol{L}^2_t$ is the energy space such that

$$\left(\boldsymbol{L}^2_t := L^2(]z(t), +\infty[_{r_*} \times S^2_\omega, r^2 F^{1/2}(r) dr_* d\omega)^4, \ \|.\|_t\right) \tag{12}$$

and

$$\boldsymbol{D}_t = -\frac{qQ}{r} + \Gamma^1 \left(\partial_{r_*} + \frac{F(r)}{r} + \frac{F^{'}(r)}{4}\right) + \sqrt{F(r)}\left(\frac{\Gamma^2}{r}(\partial_\theta + \frac{1}{2}\cot\theta) + \frac{\Gamma^3}{r\sin\theta}\partial_\varphi + \Gamma^4\right), \tag{13}$$

$$\Gamma^1 := i\gamma^0\gamma^1 = i\text{Diag}(-1,1,1,-1), \quad \Gamma^2 := i\gamma^0\gamma^2, \quad \Gamma^3 := i\gamma^0\gamma^3, \quad \Gamma^4 := -m\gamma^0, \tag{14}$$

with

$$\mathcal{D}(\boldsymbol{D}_t) = \left\{ \Psi \in \boldsymbol{L}^2_t, \ \boldsymbol{D}_t\Psi \in \boldsymbol{L}^2_t; \ \frac{\dot{z}\gamma^0 - \gamma^1}{\sqrt{1-\dot{z}^2}}\Psi(z(t), \omega) = -ie^{i\nu\gamma^5}\Psi(z(t), \omega) \right\}. \tag{15}$$

Here the Dirac matrices γ^k, satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\boldsymbol{I}_{\mathbb{R}^4}, \quad a, b = 0, .., 3, \quad \eta^{ab} = \text{Diag}(1,-1,-1,-1). \tag{16}$$

$$\gamma^0 = i\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^k = i\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \quad k = 1,2,3, \tag{17}$$

with the Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{18}$$

We introduce the following notation

$$\forall \Phi \in \boldsymbol{L}_t^2, \quad \|\Phi\|_t = \|[\Phi]_L\|, \quad [\Phi]_L(r_*,\omega) = \begin{cases} \Phi(r_*,\omega) & r_* \in]z(t), +\infty[_{r_*} \\ 0 & r_* \in \mathbb{R}\backslash]z(t), +\infty[_{r_*} \end{cases}.$$

According to proposition III.2 in [2], a unique solution $\Psi(t)$ of (9), (10) and (11) can be expressed with the propagator $\boldsymbol{U}(t,s)$:

Proposition 2.1
Given $\Psi_s \in \mathcal{D}(\boldsymbol{D}_s)$, then there exists a unique solution $[\Psi(.)]_L = [\boldsymbol{U}(.,s)\Psi_s]_L \in C^1(\mathbb{R}_t, \boldsymbol{L}^2_{BH})$ of (9), (10) and (11) such that, for all $t \in \mathbb{R}$

$$\Psi(t) \in \mathcal{D}(\boldsymbol{D}_s), \qquad \|\Psi(t)\|_t = \|\Psi_s\|_s.$$

Moreover, $\boldsymbol{U}(t,s)$ can be extended in an isometric strongly continuous propagator from $\boldsymbol{L}_s^2$ onto $\boldsymbol{L}_t^2$.

In the same way, we consider the hyperbolic problem related to (7) on $(\mathcal{M}_{\mathrm{bh}}, g)$:

$$\partial_t \Psi = i\boldsymbol{D}_{\mathrm{BH}}\Psi \tag{19}$$

$$\Psi(t=0,.) = \Psi_{\mathrm{BH}}(.) \in \boldsymbol{L}^2_{\mathrm{BH}}, \tag{20}$$

where the differential operator $\boldsymbol{D}_{\mathrm{BH}}$ has the form (13) but defined on

$$\left(\boldsymbol{L}^2_{\mathrm{BH}} := L^2(\mathbb{R}_{r_*} \times S^2_\omega, r^2 F^{1/2}(r) dr_* d\omega)^4,\ \|.\|\right). \tag{21}$$

In [13], we prove that the hamiltonian $\boldsymbol{D}_{\mathrm{BH}}$ is self adjoint with dense domain

$$\mathcal{D}(\boldsymbol{D}_{\mathrm{BH}}) = \left\{\Psi \in \boldsymbol{L}^2_{\mathrm{BH}},\ \boldsymbol{D}_{\mathrm{BH}}\Psi \in \boldsymbol{L}^2_{\mathrm{BH}}\right\}. \tag{22}$$

Hence by the spectral theorem, we have:

Proposition 2.2
The problem (19)-(20) has a unique solution $\Psi \in C^0(\mathbb{R}_t, \boldsymbol{L}^2_{BH})$ given by the strongly continuous unitary group $\boldsymbol{U}(t) := e^{it\boldsymbol{D}_{BH}}$:

$$\Psi(t) = \boldsymbol{U}(t)\Psi_{BH}, \quad \Psi(0) = \Psi_{BH}.$$

Moreover

$$\|\Psi(t)\| = \|\Psi_{BH}\|.$$

2.2 Scattering for Dirac fields by an eternal black-hole

Our result on the Hawking effect follows from a asymptotic analysis for the propagator $\boldsymbol{U}(0,T)$ as $T \to +\infty$. As the star becomes a black hole as $T \to +\infty$, we strongly use that the dynamics are simplier in vicinity of the two following asymptotic regions: $r_* \to -\infty$ (black hole horizon) and $r_* \to +\infty$ (cosmological horizon when $\Lambda > 0$ or the asymptotically flat space time when $\Lambda = 0$). This is the reason why we introduce the wave operators for the eternal charged black-hole. The existence and the asymptotic completeness for these operators are already been the subject of two previous works: [13] and [15]. To investigate the behavior of the Dirac fields near the black hole horizon (resp. cosmological horizon $\Lambda > 0$ or asymptotically flat region $\Lambda = 0$), we choose a cut function $\chi_{\leftarrow} \in C^\infty(\mathbb{R}_{r_*})$ (resp. $\chi_{\rightarrow} \in C^\infty(\mathbb{R}_{r_*})$) satisfying:

$$\exists\, a, b \in \mathbb{R},\ 0 < a < b < 1 \quad \chi_{\leftarrow}(r_*) = \begin{cases} 1 & r_* < a \\ 0 & r_* > b \end{cases}, \quad (resp.\ \chi_{\rightarrow} = 1 - \chi_{\leftarrow}). \tag{23}$$

As regards the asymptotic behavior of the fields as $r_* \to -\infty$ (resp. $r_* \to +\infty$ when $\Lambda > 0$), we compare the solution of (19) on $\boldsymbol{L}^2_{\text{BH}}$ with the solution of

$$\partial_t \Psi_{\leftarrow} = i\boldsymbol{D}_{\leftarrow} \Psi_{\leftarrow} \quad \left(resp.\ \partial_t \Psi_{\rightarrow} = \boldsymbol{D}_{\Lambda,\rightarrow} \Psi_{\rightarrow}\right) \tag{24}$$

where

$$\boldsymbol{D}_{\leftarrow} := \Gamma^1 \partial_{r_*} - \frac{qQ}{r_0} \quad \left(resp.\ \boldsymbol{D}_{\Lambda,\rightarrow} := \Gamma^1 \partial_{r_*} - \frac{qQ}{r_+}\right)$$

is self-adjoint on

$$\boldsymbol{L}^2_{\leftarrow} := L^2(\mathbb{R}_{r_*} \times S^2_\omega;\ dr_* d\omega)^4, \quad (resp.\ \boldsymbol{L}^2_{\Lambda,\rightarrow} := \boldsymbol{L}^2_{\leftarrow}, \quad \Lambda > 0),$$

with the dense domain

$$\mathcal{D}(\boldsymbol{D}_{\leftarrow}) = H^1(\mathbb{R}_{r_*}; L^2(S^2_\omega))^4 \quad \left(resp.\ \mathcal{D}(\boldsymbol{D}_{\Lambda,\rightarrow}) = H^1(\mathbb{R}_{r_*}; L^2(S^2_\omega))^4\right).$$

Since the matrix Γ^1 is diagonal, we remark that equations (24) are the shift equations according to the components. Hence, we define the subspaces of outgoing and incoming waves $\boldsymbol{L}^{2+}_{\leftarrow}$ and $\boldsymbol{L}^{2-}_{\leftarrow}$ such that $\boldsymbol{L}^2_{\leftarrow} = \boldsymbol{L}^{2+}_{\leftarrow} \oplus \boldsymbol{L}^{2-}_{\leftarrow}$,

$$\boldsymbol{L}^{2+}_{\leftarrow} := \{\Psi \in \boldsymbol{L}^2_{\leftarrow};\ \Psi_2 = \Psi_3 = 0\}, \quad \boldsymbol{L}^{2-}_{\leftarrow} := \{\Psi \in \boldsymbol{L}^2_{\leftarrow};\ \Psi_1 = \Psi_4 = 0\}, \tag{25}$$

and

$$\boldsymbol{L}^2_{\Lambda,\rightarrow} = \boldsymbol{L}^{2+}_{\Lambda,\rightarrow} \oplus \boldsymbol{L}^{2-}_{\Lambda,\rightarrow}, \quad \boldsymbol{L}^{2+}_{\Lambda,\rightarrow} := \boldsymbol{L}^{2+}_{\leftarrow}, \quad \boldsymbol{L}^{2-}_{\Lambda,\rightarrow} := \boldsymbol{L}^{2-}_{\leftarrow}. \tag{26}$$

Hence, we define the wave operators $\boldsymbol{W}^{\pm}_{\leftarrow}$ at the black-hole horizon for $\Lambda \geq 0$ and $\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}$ at the cosmological horizon when $\Lambda > 0$, by

$$\boldsymbol{W}^{\pm}_{\leftarrow} \Psi^{\pm} = \lim_{t \to \pm\infty} \boldsymbol{U}(-t) \mathcal{J}_{\leftarrow} e^{it\boldsymbol{D}_{\leftarrow}} \Psi^{\pm} \quad \text{in} \quad \boldsymbol{L}^2_{\text{BH}}, \quad \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, \quad \Lambda \geq 0 \tag{27}$$

$$\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow} \Psi^{\mp} = \lim_{t \to \pm\infty} \boldsymbol{U}(-t) \mathcal{J}_{\Lambda,\rightarrow} e^{it\boldsymbol{D}_{\Lambda,\rightarrow}} \Psi^{\mp} \quad \text{in} \quad \boldsymbol{L}^2_{\text{BH}}, \quad \Psi^{\mp} \in \boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}, \quad \Lambda > 0. \tag{28}$$

where $\mathcal{J}_{\leftarrow}$ and $\mathcal{J}_{\Lambda,\rightarrow}$ are respectively the identifying operator between $\boldsymbol{L}^2_{\leftarrow}$ and $\boldsymbol{L}^2_{\text{BH}}$ and the one between $\boldsymbol{L}^2_{\Lambda,\rightarrow}$ and $\boldsymbol{L}^2_{\text{BH}}$:

$$\mathcal{J}_{\leftarrow} : \Psi^{\pm}(r_*, \omega) \mapsto \chi_{\leftarrow}(r_*) r^{-1} F^{-1/4}(r) \Psi^{\pm}(r_*, \omega), \quad \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, \quad \Lambda \geq 0$$
$$\mathcal{J}_{\Lambda,\rightarrow} : \Psi^{\pm}(r_*, \omega) \mapsto \chi_{\rightarrow}(r_*) r^{-1} F^{-1/4}(r) \Psi^{\pm}(r_*, \omega), \quad \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\Lambda,\rightarrow}, \quad \Lambda > 0.$$

The space-time is asymptotically flat at the infinity when $\Lambda = 0$. Therefore, we compare the solutions of (9) on $\boldsymbol{L}^2_{\text{BH}}$ with the solution $\Psi_{\rightarrow}$ of the Dirac equation on Minkowski space-time with spherical coordinates $(\rho, \omega) \in \mathbb{R}^+_* \times [0, \pi] \times [0, 2\pi[$, putting $r_* = \rho > 0$ to avoid artificial long-range interactions :

$$\partial_t \Psi_{\rightarrow} = i\boldsymbol{D}_{0,\rightarrow} \Psi_{\rightarrow} \tag{29}$$

where

$$\boldsymbol{D}_{0,\rightarrow} := \Gamma^1\left(\partial_\rho + \frac{1}{\rho}\right) + \frac{\Gamma^2}{\rho}(\partial_\theta + \frac{1}{2}\cot\theta) + \frac{\Gamma^3}{\rho\sin\theta}\partial_\varphi + \Gamma^4,$$

is self-adjoint on

$$\boldsymbol{L}^2_{0,\rightarrow} := L^2(\mathbb{R}^+_\rho \times S^2_\omega;\ \rho^2 d\rho d\omega)^4$$

with the dense domain

$$\mathcal{D}(\boldsymbol{D}_{0,\rightarrow}) = H^1(\mathbb{R}^+_\rho \times S^2_\omega;\ \rho^2 d\rho d\omega)^4.$$

Since the comparison of the solution of (9) on $\boldsymbol{L}^2_{\mathrm{BH}}$ with the solution of (29) on $\boldsymbol{L}^2_{0,\rightarrow}$ involves matrix-valued long-range perturbations, it is necessary to modify the free dynamic $e^{it\boldsymbol{D}_{0,\rightarrow}}$ as in ours previous works [13] and [15]. Given $\boldsymbol{U}_{0,\rightarrow}(t)$ the Dollard-modified propagator, then we define for all $\Psi \in \boldsymbol{L}^2_{0,\rightarrow}$ the wave operator $\boldsymbol{W}^{\pm}_{0,\rightarrow}$ at infinity:

$$\boldsymbol{W}^{\pm}_{0,\rightarrow}\Psi = \lim_{t\to\pm\infty} \boldsymbol{U}(-t)\mathcal{J}_{0,\rightarrow}\boldsymbol{U}_{0,\rightarrow}(t)\Psi \quad \text{in} \quad \boldsymbol{L}^2_{\mathrm{BH}}, \tag{30}$$

where $\mathcal{J}_{0,\rightarrow}$ is the bounded identifying operator between $\boldsymbol{L}^2_{0,\rightarrow}$ and $\boldsymbol{L}^2_{\mathrm{BH}}$:

$$(\mathcal{J}_{0,\rightarrow}\Psi)(r_*,\omega) := \begin{cases} \chi_\rightarrow(r_*) r^{-1} F^{-1/4}(r) r_* \Psi(r_*,\omega) & r_* > 0 \\ 0 & r_* \le 0 \end{cases}, \ \forall \Psi \in \boldsymbol{L}^2_{0,\rightarrow}$$

Finally according to [13], [15] and [14], we state the theorem :

Theorem 2.1
The wave operators $\boldsymbol{W}^{\pm}_{\leftarrow}$, $\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}$ and $\boldsymbol{W}^{\pm}_{0,\rightarrow}$, respectively defined on $\boldsymbol{L}^{2\pm}_{\leftarrow}$, $\boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}$ and $\boldsymbol{L}^2_{0,\rightarrow}$ exist and are independent of the cut-off functions $\chi_\leftarrow$, $\chi_\rightarrow$ and $\chi_\rightarrow$ satisfying (79). Moreover:

$$Ran\left(\boldsymbol{W}^{\pm}_{\leftarrow} \oplus \boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\right) = \boldsymbol{L}^2_{BH}, \quad (\Lambda \ge 0)$$

and

$$\begin{aligned}
&\forall \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, \quad \Lambda \ge 0, \quad m \ge 0, \quad \|\boldsymbol{W}^{\pm}_{\leftarrow}\Psi^{\pm}\| = \|\Psi^{\pm}\|_{\boldsymbol{L}^2_{\leftarrow}} \\
&\forall \Psi^{\mp} \in \boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}, \quad \Lambda > 0, \quad m \ge 0, \quad \|\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\Psi^{\mp}\| = \|\Psi^{\mp}\|_{\boldsymbol{L}^2_{\Lambda,\rightarrow}} \\
&\forall \Psi \in \boldsymbol{L}^2_{0,\rightarrow}, \quad \Lambda = 0, \quad m > 0, \quad \|\boldsymbol{W}^{\pm}_{0,\rightarrow}\Psi\| = \|\Psi\|_{\boldsymbol{L}^2_{0,\rightarrow}}.
\end{aligned}$$

3 Quantum Fields

3.1 Construction of the Dirac Quantum Fields

To describe the Quantum effects of the collapse, we need to introduce the framework of the Quantum Field Theory. For a general discussion on the Quantum Field Theory in curved spacetime, we cite the following and non exhaustive list of books: [3], [8], [18], [21]. This theory are usually defined on flat space-time. In Minkowski space-time, we have a natural choice for the vacuum state: the vacuum related to the inertial observators. In this case, it is sufficient to construct a field operator which satisfies a given field equation on a Hilbert space corresponding to an inertial observator (we choose a particular Cauchy hypersurface of the space-time). Indeed, thanks to the Lorentz transformation, this construction is equivalent for all inertial observators. But in our case, we deal with a curved space-time and in general manifolds, hence we have not the equivalent Lorentz transformations and any preferential choice for the vacuum. Then, we adopt the point of view introduced by J. Dimock in [6] and [7]. In [7] and for the spin 1/2 fields, the author suggests a construction for local observables to globally hyperbolic manifolds which is independent (up to a net isomorphism) of the representation of the CAR, the choice of the spin structure and the Cauchy hypersurface.

Before to explain this construction, we define on a complex Hilbert space $(\mathfrak{H}, < .,. >_{\mathfrak{H}})$ the Fermi-Dirac Fock space describing the state with an arbitrary number of non interacting charged fermions. Given a Dirac-type equation satisfied by the field f with Hamiltonian $\mathbb{H}$ defined on $\mathfrak{H}$:

$$\partial_t f = i\mathbb{H}f. \tag{31}$$

We choose the spectral projectors P_+ and P_- such that

$$P_+ := \mathbf{1}_{]-\infty,0]}(\mathbb{H}), \quad P_- := \mathbf{1}_{[0,+\infty[}(\mathbb{H}). \tag{32}$$

Then, we introduce the Fermi-Dirac-Fock space for $(\mathfrak{H}, < .,. >_{\mathfrak{H}})$:

$$\mathfrak{F}(\mathfrak{H}) := \bigoplus_{n,m=0}^{+\infty} \mathfrak{F}^{(n,m)}, \quad \mathfrak{F}^{(n,m)}(\mathfrak{H}) := \mathfrak{F}^{(n)}(\mathfrak{H}_+) \otimes \mathfrak{F}^{(m)}(\mathfrak{H}_-), \tag{33}$$

where

$$\mathfrak{F}^{(0)}(\mathfrak{H}_+) := \mathbb{C}, \quad \mathfrak{F}^{(0)}(\mathfrak{H}_-) := \mathbb{C}, \quad \mathfrak{F}^{(n)}(\mathfrak{H}_+) := \bigwedge_{k=1}^{n} \mathfrak{H}_+, \quad \mathfrak{F}^{(m)}(\mathfrak{H}_-) := \bigwedge_{k=1}^{m} \Upsilon\mathfrak{H}_- \tag{34}$$

and

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \quad \mathfrak{H}_+ := P_+\mathfrak{H}, \quad \mathfrak{H}_- := P_-\mathfrak{H}. \tag{35}$$

Here, Υ is the charge conjugation (see [19] section 1.4.6). On $\mathfrak{F}(\mathfrak{H})$, we introduce $a(P_+f)$ and $a^*(P_+f)$ the particle annihilation and creation operators and also $b(P_-f)$, $b^*(P_-f)$ the anti-particle annihilation, creation operators. We can find the their rigorous definition in the appendix A in [2] or in the book [4]. Therefore, we define the anti-linear quantized Dirac field operator $\mathbf{\Psi}$ and its linear adjoint $\mathbf{\Psi}^*$:

$$f \in \mathfrak{H} \longmapsto \mathbf{\Psi}(f) := a(P_+f) + b^*(\Upsilon P_-f) \in \mathcal{L}(\mathfrak{F}(\mathfrak{H})), \tag{36}$$

and

$$f \in \mathfrak{H} \longmapsto \mathbf{\Psi}^*(f) := a^*(P_+f) + b(\Upsilon P_-f) \in \mathcal{L}((\mathfrak{F}(\mathfrak{H})).$$

Moreover, these operators are bounded

$$\|\mathbf{\Psi}(f)\| = \|f\|, \quad \|\mathbf{\Psi}^*(f)\| = \|f\|, \quad f \in \mathfrak{H}$$

and thanks to the classical properties of the creations and annihilation operators, it satisfies the canonical anti-commutation relations (CAR):

$$\begin{aligned}
&\mathbf{\Psi}(f)\mathbf{\Psi}(g)+\mathbf{\Psi}(g)\mathbf{\Psi}(f)=0, \quad \mathbf{\Psi}^*(f)\mathbf{\Psi}^*(g)+\mathbf{\Psi}^*(g)\mathbf{\Psi}^*(f)=0, \quad f,g\in\mathfrak{H}\\
&\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)+\mathbf{\Psi}(g)\mathbf{\Psi}^*(f)=<f,g>_{\mathfrak{H}}\mathbf{1}
\end{aligned}$$

We consider the C^*-algebra $\mathfrak{U}(\mathfrak{H})$ generated by the field operators $\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)$, with $f,g\in\mathfrak{H}$ and introduce the KMS state $\omega_{\text{KMS}}^{\delta,\sigma}$ such that for $f,g\in\mathfrak{H}$:

$$\omega_{\text{KMS}}^{\delta,\sigma}(\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)):=<\mathcal{K}_{\mu,\sigma}^{ms}(\mathbb{H})f,g>_{\mathfrak{H}}, \tag{37}$$

with, for all $x\in\mathbb{R}$

$$\mathcal{K}_{\mu,\sigma}^{ms}(x):=\mu e^{\sigma x}(1+\mu e^{\sigma x})^{-1}, \quad \mu:=e^{\sigma\delta}, \quad \sigma>0, \quad \delta\in\mathbb{R}. \tag{38}$$

On the sub-algebra $\mathfrak{U}(\mathfrak{H}_+)$ (resp. $\mathfrak{U}(\mathfrak{H}_-)$) of $\mathfrak{U}(\mathfrak{H})$, the state $\omega_{\text{KMS}}^{\delta,\sigma}$ provides a description of an thermodynamical equilibrium state for a gas noninteracting Fermi particles (resp. anti-particles) with temperature $1/\sigma>0$, chemical potential δ (resp. $-\delta$) and activity μ (resp. $1/\mu$).

Now, according to the work of J. Dimock [7], we construct the algebra of local observables on a given globally hyperbolic curved space-time $\mathcal{M}$ with a foliation by a family of Cauchy hypersurfaces Π_t :

$$\mathcal{M}=\bigcup_{t\in\mathbb{R}}\Pi_t.$$

We consider a fixed hypersurfaces Π_t and put $\mathfrak{H}=L(\Pi_t)^4$. Using the previous definition of Dirac quantum field (36), we define on $L(\Pi_t)^4$ the quantized Dirac field $\mathbf{\Psi}_a$ and $\mathfrak{U}(L(\Pi_t)^2)$ the C^*-algebra generated by $\mathbf{\Psi}_a^*(\Phi_1)\mathbf{\Psi}_a(\Phi_2)$, with $\Phi_1,\Phi_2\in L(\Pi_t)^4$. Moreover we introduce the following operator

$$S_A:\Phi\in C_0^\infty(\mathcal{M})^4\longmapsto S_A\Phi:=\int_{\mathbb{R}}P(t,s)\Phi(s)ds\in L(\Pi_t)^4, \tag{39}$$

where $P(t,s)$ is the isometric propagator from $L(\Pi_s)^4$ onto $L(\Pi_t)^4$, related to the Dirac field in $\mathcal{M}_{\text{coll}}$. Then, we define the local quantum field in $\mathcal{M}$ by the operator:

$$\mathbf{\Psi}_A:\Phi\in C_0^\infty(\mathcal{M})^4\longmapsto\mathbf{\Psi}_A(\Phi):=\mathbf{\Psi}_a(S_A\Phi), \tag{40}$$

and, for any open set $\mathcal{O}\subset\mathcal{M}$, we introduce $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\mathbf{\Psi}_A(\Phi_1)\mathbf{\Psi}_A(\Phi_2)$, $supp(\Phi_j)\subset\mathcal{O}$, $j=1,2$. Finally, we have:

$$\mathfrak{U}(\mathcal{M})=adh\left(\bigcup_{\mathcal{O}}\mathfrak{U}(\mathcal{O})\right).$$

Hence by J. Dimock [7], this construction is independent of the representation of the CAR, the choice of the spin structure contained in $P(t,s)$ and the fixed Cauchy hypersurface Π_t with $t\in\mathbb{R}$.

Now, we apply this procedure to the space time outside the collapsing star $\mathcal{M}_{\text{coll}}$ but also to the space times near the future black hole $\mathcal{M}_{\text{bh}}$ and at the infinity ($r_*\to+\infty$) $\mathcal{M}_{\text{flat}}$ or $\mathcal{M}_{\text{bh}}$, with the intention of interpreting the Hawking effect with the help of the wave operators (27), (28) and (30).

For the stationary space time $\mathcal{M}_{\text{coll}}$ we have the following foliation :

$$\mathcal{M}_{\text{coll}} = \bigcup_{t\in\mathbb{R}} \Pi_t, \quad \Pi_t := \{t\}\times]z(t), +\infty[_{r_*}\times S^2_\omega.$$

We consider Π_0, and we put

$$\mathfrak{H} := L^2(]z(0), +\infty[\times S^2_\omega, r^2 F^{1/2}(r) dr_* d\omega)^4 = \boldsymbol{L}^2_0, \quad \mathbb{H} := \boldsymbol{D}_0. \tag{41}$$

Using the previous construction, we define on $\boldsymbol{L}^2_0$ the quantized Dirac field $\boldsymbol{\Psi}_0 = \boldsymbol{\Psi}_a$ and $\mathfrak{U}(\mathfrak{H})$ the C^*-algebra generated by $\boldsymbol{\Psi}^*_0(\Phi_1)\boldsymbol{\Psi}_0(\Phi_2)$, with $\Phi_1, \Phi_2 \in \mathfrak{H}$. According to (39), we introduce $S_{\text{coll}} = S_A$ with $P(0,t) = \boldsymbol{U}(0,t)$ the propagator defined in proposition 2.1. Then, we define the local quantum field in $\mathcal{M}_{\text{coll}}$ by the operator

$$\boldsymbol{\Psi}_{\text{coll}} : \Phi \in C^\infty_0(\mathcal{M}_{\text{coll}})^4 \longmapsto \boldsymbol{\Psi}_{\text{coll}}(\Phi) := \boldsymbol{\Psi}_0(S_{\text{coll}}\Phi) \tag{42}$$

and also $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ the closed union for all open set $\mathcal{O} \subset \mathcal{M}_{\text{coll}}$ of $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\boldsymbol{\Psi}^*_{\text{coll}}(\Phi_1)\boldsymbol{\Psi}_{\text{coll}}(\Phi_2)$, $supp(\Phi_j) \subset \mathcal{O}$, $j = 1, 2$. Then, according to (37) and (41), we define on $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ a ground state $\omega_{\mathcal{M}_{\text{coll}}}$ as following:

$$\omega_{\mathcal{M}_{\text{coll}}}(\boldsymbol{\Psi}^*_{\text{coll}}(\Phi_1)\boldsymbol{\Psi}_{\text{coll}}(\Phi_2)) := \omega^{\delta_0,\sigma_0}_{\text{KMS}}(\boldsymbol{\Psi}^*_0(S_{\text{coll}}\Phi_1)\boldsymbol{\Psi}_0(S_{\text{coll}}\Phi_2)) \tag{43}$$

$$= < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)S_{\text{coll}}\Phi_1, S_{\text{coll}}\Phi_2 >_{\mathfrak{H}}, \quad \Phi_1, \Phi_2 \in \mathfrak{H} \tag{44}$$

with

$$\mu_0 := e^{\sigma_0\delta_0}, \qquad \delta_0 \in \mathbb{R}, \quad \sigma_0 > 0. \tag{45}$$

Indeed, we suppose that our star which is stationary in the past collapses in a bath of fermions and anti-fermions with temperature $\sigma_0^{-1} > 0$.

We describe the quantum field at the horizon of future back-hole. We consider the stationary space-time $\mathcal{M}_{\text{bh}}$ with the following foliation

$$\mathcal{M}_{\text{bh}} = \bigcup_{t\in\mathbb{R}} \Pi_t, \quad \Pi_t := \{t\} \times \mathbb{R}_{r_*} \times S^2_\omega.$$

By using the same procedure as above we construct $\mathfrak{U}_{\leftarrow}(\mathcal{M}_{\text{bh}})$ the closed union for all open set $\mathcal{O} \subset \mathcal{M}_{\text{bh}}$ of $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\boldsymbol{\Psi}_{\leftarrow}(\Psi_1)\boldsymbol{\Psi}^*_{\leftarrow}(\Psi_2)$, $\Phi_1, \Phi_2 \in \boldsymbol{L}^2_{\leftarrow}$ where

$$\boldsymbol{\Psi}_{\leftarrow} : \Phi \in C^\infty_0(\mathcal{M}_{\text{bh}})^4 \longmapsto \boldsymbol{\Psi}_{\leftarrow}(\Phi) := \boldsymbol{\Psi}_{\_}(S_{\leftarrow}\Phi), \tag{46}$$

and

$$S_{\leftarrow} := S_A, \quad P(0,t) := e^{-it\boldsymbol{D}_{\leftarrow}}. \tag{47}$$

Here $\boldsymbol{\Psi}_{\_}(\Phi)$ with $\Phi \in \boldsymbol{L}^2_{\leftarrow}$ is the quantum Dirac field defined on the hypersurface $\mathbb{R}_{r_*} \times S^2_\omega$. By using (37), we consider the Hawking thermal state:

$$\omega^{\delta,\sigma}_{\text{Haw}}(\boldsymbol{\Psi}^*_{\leftarrow}(\Phi_1)\boldsymbol{\Psi}_{\leftarrow}(\Phi_2)) := \omega^{\delta,\sigma}_{\text{KMS}}(\boldsymbol{\Psi}^*_{\_}(S_{\leftarrow}\Phi_1)\boldsymbol{\Psi}_{\_}(S_{\leftarrow}\Phi_2)), \quad \Phi_1, \Phi_2 \in C^\infty_0(\mathcal{M}_{\text{bh}})^4 \tag{48}$$

$$= < \mathcal{K}^{ms}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})S_{\leftarrow}\Phi_1, S_{\leftarrow}\Phi_2 >_{\boldsymbol{L}^2_{\leftarrow}}, \tag{49}$$

with

$$\mu := e^{\sigma\delta}, \quad \delta \in \mathbb{R}, \quad \sigma > 0. \tag{50}$$

Finally we introduce the quantum fields at infinity when $r_* \to +\infty$. According to Λ which is respectively positive or zero (cosmological horizon or asymptotically flat space-time), we consider the stationary space-times

$$\mathcal{M}_{\mathrm{bh}} = \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega, \quad \mathcal{M}_{\mathrm{flat}} := \mathbb{R}_t \times \mathbb{R}^+_{r_*} \times S^2_\omega.$$

As above, using the Fermi-Dirac Fock quantization on $\mathbb{R}_{r_*} \times S^2_\omega$ or $\mathbb{R}^+_{r_*} \times S^2_\omega$, we define the fields $\boldsymbol{\Psi}_{\Lambda,+}(\Phi_1)$ with $\Phi_1 \in \boldsymbol{L}^2_{\Lambda,\to}$ or $\boldsymbol{\Psi}_{0,+}(\Phi_1)$ with $\Phi_1 \in \boldsymbol{L}^2_{0,\to}$. Hence, we construct $\mathfrak{U}_\to(\mathcal{M}_{\mathrm{bh}})$ and $\mathfrak{U}_\to(\mathcal{M}_{\mathrm{flat}})$. The algebra $\mathfrak{U}_\to(\mathcal{M}_{\mathrm{bh}})$ is the closed union for all open set $\mathcal{O} \subset \mathcal{M}_{\mathrm{bh}}$ of the C^*-algebras $\mathfrak{U}_\to(\mathcal{O})$ generated by $\boldsymbol{\Psi}^*_{\Lambda,\to}(\Phi_1)\boldsymbol{\Psi}_{\Lambda,\to}(\Phi_1)$ with $\Phi_1,\ \Phi_2 \in \boldsymbol{L}^2_{\Lambda,\to}$

$$\boldsymbol{\Psi}_{\Lambda,\to} : \Phi \in C_0^\infty(\mathcal{M}_{\mathrm{bh}})^4 \longmapsto \boldsymbol{\Psi}_{\Lambda,\to}(\Phi) := \boldsymbol{\Psi}_{\Lambda,+}(S_{\Lambda,\to}\Phi), \quad \Lambda > 0 \tag{51}$$

and

$$S_{\Lambda,\to} := S_A, \quad P(0,t) := e^{-it\boldsymbol{D}_{\Lambda,\to}}, \quad \Lambda > 0. \tag{52}$$

As to the algebra $\mathfrak{U}_\to(\mathcal{M}_{\mathrm{flat}})$, it is the closed union for all $\mathcal{O} \subset \mathcal{M}_{\mathrm{flat}}$ of the C^*-algebras $\mathfrak{U}_\to(\mathcal{O})$ generated by $\boldsymbol{\Psi}^*_{0,\to}(\Phi_1)\boldsymbol{\Psi}_{0,\to}(\Phi_1)$ with $\Phi_1, \Phi_2 \in \boldsymbol{L}^2_{0,\to}$,

$$\boldsymbol{\Psi}_{0,\to} : \Phi \in C_0^\infty(\mathcal{M}_{\mathrm{flat}})^4 \longmapsto \boldsymbol{\Psi}_{0,\to}(\Phi) := \boldsymbol{\Psi}_{0,+}(S_{0,\to}\Phi) \tag{53}$$

and

$$S_{0,\to} := S_A, \quad P(0,t) := \boldsymbol{U}_{0,\to}(-t), \tag{54}$$

where $\boldsymbol{U}_{0,\to}$ is the Dollard-modified propagator. With (37), the thermal states on each algebras $\mathfrak{U}_\to(\mathcal{M}_{\mathrm{bh}})$ and $\mathfrak{U}_\to(\mathcal{M}_{\mathrm{flat}})$ are given by

$$\forall \Phi_1, \Phi_2 \in C_0^\infty(\mathcal{M}_{\mathrm{bh}}), \quad \omega^{\delta_0,\sigma_0}_{\mathrm{KMS}}(\boldsymbol{\Psi}^*_{\Lambda,\to}(\Phi_1)\boldsymbol{\Psi}_{\Lambda,\to}(\Phi_1)) = < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\to})S_{\Lambda,\to}\Phi_1, S_{\Lambda,\to}\Phi_2 >_{\boldsymbol{L}^2_{\Lambda,\to}},$$

with $\Lambda > 0$, and

$$\forall \Phi_1, \Phi_2 \in C_0^\infty(\mathcal{M}_{\mathrm{flat}}), \quad \omega^{\delta_0,\sigma_0}_{\mathrm{KMS}}(\boldsymbol{\Psi}^*_{0,\to}(\Phi_1)\boldsymbol{\Psi}_{0,\to}(\Phi_1)) = < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{0,\to})S_{0,\to}\Phi_1, S_{0,\to}\Phi_2 >_{\boldsymbol{L}^2_{0,\to}}.$$

3.2 Hawking effect

The state

$$\omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi_1)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi_2)), \quad \Phi_j \in C_0^\infty(\mathcal{M}_{\mathrm{coll}})^4, \quad j = 1, 2,$$

gives the informations about the quantum fluctuations in a region of $\mathcal{M}_{\mathrm{coll}}$. But, we are interested in the investigation of this previous state at last moment of gravitational collapse when the detector is fixed with the respect to the variables (r_*, ω). As this collapsing star becomes a black hole, the detector at the rest receives the informations from the creation of the black hole when this proper time $t = \infty$. Hence, we put

$$\Phi^T_j(t, r_*, \omega) := \Phi_j(t - T, r_*, \omega), \quad \Phi_j \in C_0^\infty(\mathcal{M}_{\mathrm{coll}})^4, \quad j = 1, 2,$$

and state the main theorem about the behavior of $\omega_{\mathcal{M}_{\mathrm{coll}}}$ at the last time of the collapse :

Theorem 3.1
Given $\Phi_j \in C_0^\infty(\mathcal{M}_{coll})^4, \quad j = 1, 2,$ *then we have for* $\Lambda \geq 0$,

$$\lim_{T \to +\infty} \omega_{\mathcal{M}_{coll}}(\mathbf{\Psi}^*_{coll}(\Phi_1^T)\mathbf{\Psi}_{coll}(\Phi_2^T)) = \omega^{\delta,\sigma}_{Haw}(\mathbf{\Psi}^*_{\leftarrow}(\mathbf{\Omega}^-_{\leftarrow}\Phi_1)\mathbf{\Psi}_{\leftarrow}(\mathbf{\Omega}^-_{\leftarrow}\Phi_2))$$
$$+\, \omega^{\delta_0,\sigma_0}_{KMS}(\mathbf{\Psi}^*_{\Lambda,\rightarrow}(\mathbf{\Omega}^-_{\Lambda,\rightarrow}\Phi_1)\mathbf{\Psi}_{\Lambda,\rightarrow}(\mathbf{\Omega}^-_{\Lambda,\rightarrow}\Phi_2)),$$

with

$$T_{Haw} = \frac{1}{\sigma} = \frac{2\pi}{\kappa_0}, \quad \delta = \frac{qQ}{r_0}.$$

Let us interpret the previous theorem.We know that the state $\omega_{\mathcal{M}_{\mathrm{coll}}}$ represents the response of a detector at the rest in Schwarzschild variables at time T. This detector is initially put in the state that corresponds for a static observer to a fermionic gas, where the particles does not interact between themselves and defined by the constants of temperature $\sigma_0 > 0$ and chemical potential δ_0.

As $T = +\infty$, the detector measures the fluctuation of the quantum fluctuations related to $\omega_{\mathcal{M}_{\mathrm{coll}}}$ when the star becomes a black hole. In this situation, the detector measures two types of informations: about the fields coming from the past infinity (and falling into the black hole) and about the fields coming from the the future horizon of the black hole (going to the future infinity).

Since the state $\omega^{\delta_0,\sigma_0}_{\mathrm{KMS}}$ contains the wave operators $\mathbf{\Omega}^-_{\Lambda,\rightarrow}$ in its definition, $\omega^{\delta_0,\sigma_0}_{\mathrm{KMS}}$ gives the information about the fields of the first type. It means that the detector measure a quantum fluctuation coming from the past infinity which is interpreted by a static observer as a flux of particles with the same characteristics that the initial ground state.

In the same way, since $\omega^{\delta,\sigma}_{\mathrm{Haw}}$ contains the wave operators $\mathbf{\Omega}^-_{\leftarrow}$ in its definition, this state gives the informations about the fields coming from the future black hole horizon. Indeed, the detector measures the emergence of the thermal state with temperature

$$T_{\mathrm{Haw}} = \frac{1}{\sigma} = \frac{2\pi}{\kappa_0}$$

which is interpreted by a static observer as flux of particles and anti-particles with charge density

$$\rho_{\mathrm{Haw}} := \frac{1}{\pi} q\delta = \frac{q^2 Q}{\pi r_0}.$$

We remark that the result is independent of the story of the collapse, the boundary condition (the characteristic of the star surface) and also the ground state since we proved the same result in [14] by supposing that the ground state is Boulware type in the past. This is a *no hair* result.

Moreover, the previous theorem is valid when $\Lambda \geq 0$. When $\Lambda > 0$, we consider the DeSitter-Reissner-Nordstrøm space time outside the star before and during the collapse. Let us recall that this curved space time has a cosmological horizon at infinity. In this case, G. W. Gibbons and S. W. Hawking have proved in [10] that an observer following any time like geodesics measures an isotropic background of thermal radiation coming from the past cosmological horizon with the (Gibbons-Hawking) temperature

$$T_{\mathrm{GH}} = \frac{2\pi}{\kappa_+}.$$

Here κ_+ is the surface gravity at the cosmological horizon defined in (2). Hence, a static observer interprets this radiation as flux of particles coming from the past cosmological horizon with temperature $T_{\mathrm{GH}} = \sigma_{GH}^{-1}$ and chemical potential δ_{GH}. Hence, we define the ground state $\omega_{\mathcal{M}_{\mathrm{coll}}}$ outside the collapsing star. On $\mathfrak{U}(\mathcal{M}_{\mathrm{coll}})$ and for all $\Phi_1, \Phi_2 \in \boldsymbol{L}_0^2$ we have

$$\begin{aligned}
\omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}_{\mathrm{coll}}^*(\Phi_1)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi_2)) &:= \omega_{\mathrm{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}_0^*(S_{\mathrm{coll}}\Phi_1)\boldsymbol{\Psi}_0(S_{\mathrm{coll}}\Phi_2)) \\
&=< \mathcal{K}_{\mu_0,\sigma_0}^{ms}(\boldsymbol{D}_0)S_{\mathrm{coll}}\Phi_1, S_{\mathrm{coll}}\Phi_2 >_{\boldsymbol{L}_0^2}, \\
&=< \boldsymbol{W}_{\Lambda,D}^{-}\, \mathcal{K}_{\mu_0,\sigma_0}^{ms}(\boldsymbol{D}_0)S_{\mathrm{coll}}\Phi_1, \boldsymbol{W}_{\Lambda,D}^{-}\, S_{\mathrm{coll}}\Phi_2 >_{\boldsymbol{L}_{\Lambda,\rightarrow}^2}, \\
&=< \mathcal{K}_{\mu_0,\sigma_0}^{ms}(\boldsymbol{D}_{\Lambda,\rightarrow})S_{\Lambda,\rightarrow}\boldsymbol{W}_{\Lambda,D}^{-}\,\Phi_1, S_{\Lambda,\rightarrow}\boldsymbol{W}_{\Lambda,D}^{-}\,\Phi_2 >_{\boldsymbol{L}_{\Lambda,\rightarrow}^2} \\
&= \omega_{\mathrm{KMS}}^{\delta_0,\sigma_0}(\boldsymbol{\Psi}_{\Lambda,\rightarrow}^*(\boldsymbol{W}_{\Lambda,D}^{-}\,\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\boldsymbol{W}_{\Lambda,D}^{-}\,\Phi_2)),
\end{aligned}$$

where $\boldsymbol{W}_{\Lambda,D}^{-}$ is the wave operator linking the dynamic outside the star before the beginning of the collapse and the free dynamic at the past cosmological horizon (see (80), (142) and (143) for the definition). Hence, in the case of cosmological model with a positive cosmological constant, the only physically relevant choice for the σ_0 and δ_0 is

$$\sigma_0 = \sigma_{GH} = T_{\mathrm{GH}}^{-1} = \frac{\kappa_+}{2\pi}, \quad \delta_0 = \delta_{GH}.$$

4 Proof of theorem 3.1.

This section is devoted to the proof of theorem 3.1. In other to demonstrate this previous theorem section 4.3, we prove the following sharp result:

Theorem 4.1
Given $f \in \boldsymbol{L}_{BH}^2$, if $\Lambda \geq 0$, then

$$\begin{aligned}
\lim_{T\to+\infty} < \mathcal{K}_{\mu_0,\sigma_0}^{ms}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)f, \boldsymbol{U}(0,T)f >_{\mathfrak{H}} &=< \mathcal{K}_{\mu_0,\sigma_0}^{ms}(\boldsymbol{D}_{\Lambda,\rightarrow})\boldsymbol{\Omega}_{\Lambda,\rightarrow}^{-}\, f, \boldsymbol{\Omega}_{\Lambda,\rightarrow}^{-}\, f >_{\boldsymbol{L}_{\Lambda,\rightarrow}^2} \\
&+ < \mathcal{K}_{\mu,\sigma}^{ms}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}_{\leftarrow}^{-} f, \boldsymbol{\Omega}_{\leftarrow}^{-} f >_{\boldsymbol{L}_{\leftarrow}^2} \qquad (55)
\end{aligned}$$

with

$$\mu = e^{\sigma\delta}, \quad \delta := \frac{qQ}{r_0} \quad \sigma = \frac{2\pi}{\kappa_0}, \quad \boldsymbol{\Omega}_{\leftarrow}^{-} := \left(\boldsymbol{W}_{\leftarrow}^{-}\right)^*, \quad \boldsymbol{\Omega}_{\Lambda,\rightarrow}^{-} := \left(\boldsymbol{W}_{\Lambda,\rightarrow}^{-}\right)^*, \quad \boldsymbol{\Omega}_{0,\rightarrow}^{-} := \left(\boldsymbol{W}_{0,\rightarrow}^{-}\right)^*,$$

where $\boldsymbol{W}_{\leftarrow}^{-}$, $\boldsymbol{W}_{\Lambda,\rightarrow}^{-}$, $\boldsymbol{W}_{0,\rightarrow}^{-}$ are the wave operators respectively defined in (27), (28) and (30).

To prove the limit (55), we use the spherical symmetry property of the geometrical framework. Indeed, we introduce the spin-weighted harmonics to reduce our study to a family of one dimensional problems. This is the purpose of the next section.

4.1 Reduction to a simplier problems thanks to the spherical symmetry.

Given $Y_{\pm\frac{1}{2},n}^l$ the spin-weighted harmonics (see [9], [13]) such that the families

$$\left\{Y_{\frac{1}{2},n}^l;\ (l,n) \in \mathcal{I}\right\}, \quad \left\{Y_{-\frac{1}{2},n}^l;\ (l,n) \in \mathcal{I}\right\}, \quad \mathcal{I} := \left\{(l,n) :\ l - \frac{1}{2} \in \mathbb{N},\ l - |n| \in \mathbb{N}\right\},$$

form a Hilbert basis of $L^2(S^2_\omega)$ and each Y^l_{sn}, $s=\pm 1/2$ satisfies the recurrence relations,

$$\partial_\theta Y^l_{sn}(\omega) \mp \frac{n - s\cos\theta}{\sin\theta} Y^l_{sn}(\omega) = \left| \begin{array}{l} -i\sqrt{(l\pm s)(l\mp s+1)}Y^l_{s\mp 1,n}(\omega), \quad \pm l > -s. \\ 0, \quad l = \mp s. \end{array} \right. , \tag{56}$$

$$\partial_\varphi Y^l_{sn}(\omega) = -in Y^l_{sn}(\omega). \tag{57}$$

Afterwards, we introduce the following Hilbert spaces:

$$\left(L^2_t := L^2(]z(t), +\infty[_{r_*},\ dr_*)^4,\ \|.\|_t\right), \quad 0 \le t \tag{58}$$

$$\left(L^2_{\mathbb{R}} := L^2(\mathbb{R}_{r_*},\ dr_*)^4,\ \|.\|\right), \tag{59}$$

$$L^2_{\text{BH}} := L^2(\mathbb{R}_{r_*},\ r^2 F^{1/2}(r) dr_*)^4 = \mathcal{P}_r L^2_{\mathbb{R}}, \tag{60}$$

with

$$\mathcal{P}_r : \Psi \mapsto r^{-1} F^{-1/4} \Psi. \tag{61}$$

So, we express $\boldsymbol{L}^2_t$ and $\boldsymbol{L}^2_{\text{BH}}$ as a direct sum:

$$\boldsymbol{L}^2_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_t, \qquad \boldsymbol{L}^2_{\text{BH}} = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_{\mathbb{R}}. \tag{62}$$

where,

$$\mathcal{E}^\nu_{ln} : \Psi_{ln} \in L^2_t \mapsto e^{-i\frac{\nu}{2}\gamma^5} \mathcal{P}_r \Psi_{ln} \otimes_4 Y_{ln} \in \boldsymbol{L}^2_t \tag{63}$$

with

$$\begin{aligned} &v \otimes_4 u := (u_1 v_1, u_2 v_2, u_3 v_3, u_4 v_4), \quad \forall u, v \in \mathbb{C}^4, \\ &Y_{ln} := \left(Y^l_{-\frac{1}{2},n}, Y^l_{\frac{1}{2},n}, Y^l_{-\frac{1}{2},n}, Y^l_{\frac{1}{2},n}\right). \end{aligned} \tag{64}$$

Defining the following restriction operator $\mathcal{R}^\nu_{ln}$ such that

$$\mathcal{R}^\nu_{ln} : \Psi \in \boldsymbol{L}^2_t \mapsto e^{i\frac{\nu}{2}\gamma^5} \mathcal{P}_r^{-1} \Psi_{ln} \in L^2_t, \quad \Psi_{ln} := < \Psi, Y_{ln} > \tag{65}$$

and using (56), (57) for $s = \pm 1/2$, we obtain the following decompositions:

$$\boldsymbol{D}_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} D_{V_{l,\nu},t} \mathcal{R}^\nu_{ln} - \frac{qQ}{r_0}, \tag{66}$$

$$D_{V_{l,\nu},t} := \Gamma^1 \partial_{r_*} + V_{l,\nu}, \quad V_{l,\nu} = qQ\left(\frac{1}{r_0} - \frac{1}{r}\right) - \sqrt{F(r)}\left(mA_\nu + \frac{i}{r}\Gamma^2(l+1/2)\right), \tag{67}$$

$$A_\nu := \begin{pmatrix} 0 & a_\nu \\ \bar{a_\nu} & 0 \end{pmatrix}, \quad a_\nu := \operatorname{diag}(ie^{i\nu}, ie^{i\nu}), \quad Z(t) = \sqrt{\frac{1 - \dot{z}(t)}{1 + \dot{z}(t)}}, \tag{68}$$

$$\begin{aligned} \mathcal{D}(D_{V_{l,\nu},t}) = \{\Psi \in L^2_t;\ &D_{V_{l,\nu},t}\Psi \in L^2_t, \\ &Z(t)\Psi_2(z(t)) = \Psi_4(z(t)),\ \Psi_1(z(t)) = -Z(t)\Psi_3(z(t))\}. \end{aligned} \tag{69}$$

For $\Phi \in L^2(B,\ dr_*)^4$, $B \subset \mathbb{R}$, we define a L^2-extension such that

$$\|\Phi\|_{L^2(B,\ dr_*)^4} = \|[\Phi]_L\|, \quad [\Phi]_L(r_*) := \begin{cases} \Phi(r_*) & r_* \in B \\ 0 & r_* \in \mathbb{R}\setminus B \end{cases}.$$

In the same way, we introduce

$$0 \leq t, \quad H^1_t := \left\{\Phi \in L^2_t,\ \partial_{r_*}\Phi \in L^2_t\right\}, \quad H^1_{\mathbb{R}} := \left\{\Phi \in L^2_{\mathbb{R}},\ \partial_{r_*}\Phi \in L^2_{\mathbb{R}}\right\},$$

and a H^1-extension such that for $\Phi \in H^1_t$ we have,

$$[\Phi]_H \in H^1_{\mathbb{R}}, \quad [\Phi]_H(r_*) := \begin{cases} \Phi(r_*) & r_* \in]z(t), +\infty[_{r_*} \\ \Phi(2z(t) - r_*) & r_* \in \mathbb{R}\backslash]z(t), +\infty[_{r_*} \end{cases}.$$

For the dynamic $D_{V_{l,\nu},t}$, we set proposition VI.2 in [2] which gives a unique solution expressed with the propagator $U_{V_{l,\nu}}(t,s)$ of:

$$\partial_t \Phi = iD_{V_{l,\nu},t}\Phi, \quad t \in \mathbb{R}, \quad r_* > z(t), \tag{70}$$

$$\Phi_4(t, z(t)) = Z(t)\Phi_2(t, z(t)), \quad \Phi_1(t, z(t)) = -Z(t)\Phi_3(t, z(t)), \tag{71}$$

$$\Phi(t = s, .) = \Phi_s(.) \in L^2_s. \tag{72}$$

Proposition 4.1
If $\Phi_s \in \mathcal{D}(D_{V_{l,\nu},s})$, then there exists a unique solution

$$[\Phi(.)]_H = [U_{V_{l,\nu}}(., s)\Phi_s]_H \in C^1(\mathbb{R}_t, L^2_{\mathbb{R}}) \cap C^0(\mathbb{R}_t, H^1_{\mathbb{R}})$$

of (70), (71) and (72) with

$$\Phi(t) \in \mathcal{D}(D_{V_{l,\nu},t}).$$

Moreover,

$$\|\Phi(t)\|_t = \|\Phi_s\|_s \tag{73}$$

and $U_{V_{l,\nu}}(t,s)$ can be extended in an isometric strongly continuous propagator from L^2_s onto L^2_t.

The operators (63) and (65) are very useful to express $\boldsymbol{U}(t,s)$ defined in proposition (2.1) with the help of $U_{V_{l,\nu}}(t,s)$:

$$\boldsymbol{U}(t,s) = e^{i(s-t)\frac{qQ}{r_0}} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} U_{V_{l,\nu}}(t,s)\mathcal{R}^\nu_{ln} : \boldsymbol{L}^2_s = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_s \to \boldsymbol{L}^2_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} L^2_t. \tag{74}$$

Given a potential $V \in L^\infty(\mathbb{R}_{r_*})$ and an interval $B := (a, +\infty)$ or $B := (-\infty, a)$ and $V \in L^\infty(\mathbb{R}_{r_*})$, then, we define on $L^2(B)^4$ the self-adjoint operator $D_{V,B}$ with the dense domain $\mathcal{D}(D_{V,B})$ such that

$$D_{V,B} = \Gamma^1\partial_{r_*} + V, \tag{75}$$

$$\mathcal{D}(D_{V,B}) = \left\{\Phi \in L^2(B)^4;\ D_{V,B}\Phi \in L^2(B)^4,\ r_* \in \partial B \Rightarrow \vec{n}\gamma^1\Phi(r_*) = i\Phi(r_*)\right\}, \tag{76}$$

where Γ^1 is given by (14) and $\vec{n}$ is the outgoing normal of B. Using Kato-Rellich and spectral theorem, it is easy to find an unique solution of

$$\partial_t\Phi = iD_{V,B}\Phi, \quad \Phi(0) = \Psi_0. \tag{77}$$

using the propagator $U_{V,B}(t)$:

Proposition 4.2
Given $\Phi_0 \in \mathcal{D}(D_{V,B})$, *then there exists a unique solution*

$$\Phi(.) = U_{V,B}(.)\Phi_0 \in C^0(\mathbb{R}_t, \mathcal{D}(D_{V,B})) \cap C^1(\mathbb{R}_t, L^2(B)^4)$$

and

$$\|\Phi(t)\| = \|\Phi_0\|.$$

Moreover, $U_{V,B}(t)$ *can be extended, by density and continuity, in strongly unitary group on* $L^2(B)^4$.

Thus, we can express the propagator $\boldsymbol{U}(t)$ defined in proposition 2.2 with the help of $U_{V,B}(t)$ and the operators (63) and (65):

$$\boldsymbol{U}(t) = e^{-it\frac{qQ}{r_0}} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^{\nu} U_{V_{l,\nu},\mathbb{R}}(t) \mathcal{R}_{ln}^{\nu}. \tag{78}$$

Now, we introduce the useful wave operators for the next part. We choose a cut-off function $\chi \in C^\infty(\mathbb{R}_{r_*})$, such that

$$\exists\; a, b \in \mathbb{R}, \quad -\infty < a < b < +\infty \quad \chi(r_*) = \begin{cases} 1 & r_* < a \\ 0 & r_* > b \end{cases}, \tag{79}$$

and the subspaces $L^{2+}_{\mathbb{R}}$ and $L^{2-}_{\mathbb{R}}$ of $L^2_{\mathbb{R}}$ with the following properties :

$$L^{2+}_{\mathbb{R}} = \left\{\Phi \in L^2_{\mathbb{R}};\ \Phi_2 \equiv \Phi_3 \equiv 0\right\}, \quad L^{2-}_{\mathbb{R}} = \left\{\Phi \in L^2_{\mathbb{R}};\ \Phi_1 \equiv \Phi_4 \equiv 0\right\}.$$

Hence, we state the lemma:

Lemma 4.1
Given $V = V_{l,\nu}$ *to simplify the notation. The wave operators*

$$\begin{aligned} W^{\pm}_{0,\mathbb{R}} &= s - \lim_{t\to\pm\infty} U_{0,\mathbb{R}}(-t)\chi U_{V,\mathbb{R}}(t), \quad in \quad L^2_{\mathbb{R}} \\ W^{\pm}_{V,[z(0),+\infty[} &= s - \lim_{t\to\pm\infty} U_{V,[z(0),+\infty[}(-t)(1-\chi)U_{V,\mathbb{R}}(t) \quad in \quad L^2_0 \end{aligned} \tag{80}$$

exist and are independent of χ *satisfying (79). Moreover*

$$Ran\left(W^{\pm}_{0,\mathbb{R}}\right) = L^{2\pm}_{\mathbb{R}}, \quad Ran\left(W^{\pm}_{V,[z(0),+\infty[}\right) = P_{ac}\left(D_{V,[z(0),+\infty[}\right) L^2_0 \tag{81}$$

where $P_{ac}\left(D_{V,[z(0),+\infty[}\right)$ *is the projector on the absolutely continuous subspace of* $D_{V,[z(0),+\infty[}$.

Proof: See lemma 6.3 in [14]. ■

By using the operators (63) and (65), we easily remark that

$$\mathcal{P}_r\left(\boldsymbol{W}^{-}_{\leftarrow}\right)^* = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln} W^{-,l}_{0,\mathbb{R}} \mathcal{R}^{\nu}_{ln}. \tag{82}$$

4.2 Proof of theorem 4.1

Firstly, we describe the main ideas of the demonstration. Our proof uses some results from some previous works: the sharp study of the backward propagator $\boldsymbol{U}(0,T)$ [14], the scattering theory in the eternal charged black hole [13, 15, 14]. With operators (63) and (65) we obtain the important relation:

$$\mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\boldsymbol{U}(0,T) = e^{iT\delta} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln}\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_{l,\nu},0})U_{V_{l,\nu}}(0,T)\mathcal{R}^{\nu}_{ln}, \quad \delta := \frac{qQ}{r_0},$$

Hence, using the spherical invariance, we reduce our study to a one dimensional problem *i.e.* the study of $\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_{l,\nu},0})U_{V_{l,\nu}}(0,T)$ as $T\to+\infty$. Now, we forget subscripts ln and ν to simplify the notations. As in [14], we split our investigation in two part thanks to the following cut off function $\mathcal{J}\in C^\infty(\mathbb{R}_{r_*})$ satisfying

$$\exists\, a,b\in\mathbb{R},\ 0<a<b<1 \quad \mathcal{J}(r_*) = \begin{cases} 1 & r_*<a \\ 0 & r_*>b \end{cases}. \tag{83}$$

Henceforth, we have

$$\mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})U_V(0,T) = \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T) + \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_V(0,T), \tag{84}$$

where the two last term are asymptotically orthogonal as $T\to+\infty$. Far from the star and thanks to the hyperbolicity, we have:

$$\mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_V(0,T) = \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_{V,\mathbb{R}}(-T),$$

where $U_{V,\mathbb{R}}$ is defined by proposition 4.2. Since this last propagator is straight linked with $\boldsymbol{U}(t)$ by formula (78), the scattering theory in the eternal charged black hole is very useful to conclude. Near the star, we prove that

$$\mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f \sim \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f, \quad T\to+\infty, \quad f\in L^2_{\mathbb{R}}. \tag{85}$$

This relation requires some technical lemmas, mainly of compactness. Thus, the weak convergence of $\mathcal{J}U_V(0,T)$ as $T\to+\infty$ is an important property to obtain the result. To conclude the proof, we use a result from a previous work [14]:

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f \sim < \mathcal{K}^{ms}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}}f, W^-_{0,\mathbb{R}}f >_{L^2_{\mathbb{R}}}, \quad T\to+\infty, \quad f\in L^2_{\mathbb{R}}, \tag{86}$$

seeing that the wave operator $W^-_{0,\mathbb{R}}$ is linked with $\boldsymbol{W}^-_{\leftarrow}$ by formula (82).

We introduce some notations :

$$D_{V,0} := D_{V,[z(0),+\infty[}, \quad L^2_0 := L^2([z(0),+\infty[_{r_*},\ dr_*)^4. \tag{87}$$

For $g := (g_1,g_2,g_3,g_4)\in L^2_{\mathbb{R}}$,

$$g^T(.) := g(.-T), \quad T\geq 0$$

and

$$G(r_*) := \frac{1}{\sqrt{-\kappa_0 r_*}}\,{}^t(-g_3,0,0,g_2)\left(-\frac{1}{2\kappa_0}\ln(-r_*) + \frac{1}{2\kappa_0}\ln(C_{\kappa_0}) + \frac{1}{2}\right), \quad r_*<0,$$

with $C_{\kappa_0}>0$. To obtain relation (85), we set and proof some lemmas. For this, we use the notations introduce by formulas (66) (67), (75) (76) and propositions 4.1 and 4.2.

Lemma 4.2
Given ${}^t(0, g_2, g_3, 0) \in C_0^\infty(\mathbb{R})^4$, *then for* $\Lambda \geq 0$ *:*

$$\lim_{T\to+\infty} < (\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{_{0,\mathbb{R}}}) - 1)\mathbf{1}_{[0,+\infty[}(D_{_{0,\mathbb{R}}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}} = 0, \tag{88}$$

$$\lim_{T\to+\infty} < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{_{0,\mathbb{R}}})\mathbf{1}_{]-\infty,0]}(D_{_{0,\mathbb{R}}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}} = 0, \tag{89}$$

Proof:
We remark that

$$\left|\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 = 4\kappa_0 B(T)|\theta(B(T)\xi)|^2, \tag{90}$$

$$\theta(B(T)\xi) := \int_{\mathbb{R}} e^{-\kappa_0 y} e^{i\xi B(T)e^{-2\kappa_0 y}} g(y)dy,\ B(T) := C_{\kappa_0} e^{-2\kappa_0 T + \kappa_0}. \tag{91}$$

Moreover, since $G_2^T \equiv G_3^T \equiv 0$, we have for $C_1 > 0$

$$\begin{aligned}\left\|\left(\mathcal{K}^{ms}_{\mu_0,\nu_0}(D_{_{0,\mathbb{R}}}) - 1\right)\mathbf{1}_{[0,+\infty[}(D_{_{0,\mathbb{R}}})[G^T]_L\right\|^2 &= C_1 \int_0^{+\infty} \left|\left(\mathcal{K}^{ms}_{\mu_0,\nu_0}(\xi) - 1\right)\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 d\xi, \\ &= C_1 \int_0^{+\infty} \left|\mathcal{K}^{ms}_{\mu_0,\nu_0}\left(\frac{\eta}{B(T)}\right) - 1\right|^2 |\theta(\eta)|^2\, d\eta.\end{aligned}$$

Since $\eta \geq 0$ and $\|[G^T]_L\| \leq \|g\|$, then $\mathcal{K}^{ms}_{\mu_0,\nu_0}\left(\frac{\eta}{B(T)}\right) - 1 \to 0$ as $T \to +\infty$. By the Cauchy-Schwartz inequality and the Lebesgue theorem, we obtain the limit (88). For the limit (89), we have

$$\begin{aligned}\left\|\mathcal{K}^{ms}_{\mu_0,\nu_0}(D_{_{0,\mathbb{R}}})\mathbf{1}_{]-\infty,0]}(D_{_{0,\mathbb{R}}})[G^T]_L\right\|^2 &= C_2 \int_{-\infty}^0 \left|\mathcal{K}^{ms}_{\mu_0,\nu_0}(\xi)\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 d\xi, \quad C_2 > 0, \\ &= C_2 \int_{-\infty}^0 \left|\mathcal{K}^{ms}_{\mu_0,\nu_0}\left(\frac{\eta}{B(T)}\right)\right|^2 |\theta(\eta)|^2\, d\eta.\end{aligned}$$

Since $\eta \leq 0$ then $\mathcal{K}^{ms}_{\mu_0,\nu_0}\left(\frac{\eta}{B(T)}\right) \to 0$ and we conclude as above. ∎

Lemma 4.3
For $\varsigma < 0$ $(\Lambda = 0)$, *we have for* $z \in \mathbb{C} \setminus \mathbb{R}$

$$\left\|\left(D_{_{\varsigma A_\nu,]-\infty, z(0)]}} \oplus D_{_{\varsigma A_\nu, [z(0), +\infty[}} - z\right)^{-1} - \left(D_{_{\varsigma A_\nu, \mathbb{R}}} - z\right)^{-1}\right\| \leq \frac{C}{|\Im z|^2}, \quad C > 0. \tag{92}$$

Proof:
For $f = (f_1, f_2, f_3, f_4) \in L^2_{\mathbb{R}}$ and $\Im z > 0$ we have

$$\left((D_{_{0,\mathbb{R}}} - z)^{-1} f\right)(r_*) = u(r_*), \quad r_* \in \mathbb{R} \tag{93}$$

with

$$j = 1, 4 \ \Rightarrow\ u_j(r_*) = -i \int_{r_*}^{+\infty} e^{-iz(r_* - y)} f_j(y)dy, \tag{94}$$

$$j = 2, 3 \ \Rightarrow\ u_j(r_*) = -i \int_{-\infty}^{r_*} e^{iz(r_* - y)} f_j(y)dy. \tag{95}$$

In the same time, we have also :

$$\left((D_{0,[z(0),+\infty[} - z)^{-1}f\right)(r_*) = u^+(r_*), \quad r_* \in [z(0), +\infty[\tag{96}$$

with

$$\begin{aligned}
u_1^+(r_*) &= -i\int_{r_*}^{+\infty} e^{-iz(r_*-y)} f_1(y)dy, \qquad u_4^+(r_*) = -i\int_{-\infty}^{r_*} e^{-iz(r_*-y)} f_4(y)dy, \\
u_2^+(r_*) &= -i\int_{z(0)}^{r_*} e^{iz(r_*-y)} f_2(y)dy - ie^{iz(r_*-z(0))}\int_{z(0)}^{+\infty} e^{izy} f_4(y)dy, \\
u_3^+(r_*) &= -i\int_{z(0)}^{r_*} e^{iz(r_*-y)} f_3(y)dy + ie^{iz(r_*-z(0))}\int_{z(0)}^{+\infty} e^{izy} f_1(y)dy.
\end{aligned}$$

and

$$\left((D_{0,]-\infty,z(0)]} - z)^{-1}f\right)(r_*) = u^-(r_*), \quad r_* \in]-\infty, z(0)] \tag{97}$$

with

$$\begin{aligned}
u_2^-(r_*) &= -i\int_{-\infty}^{r_*} e^{iz(r_*-y)} f_2(y)dy, \qquad u_3^+(r_*) = -i\int_{-\infty}^{r_*} e^{iz(r_*-y)} f_3(y)dy, \\
u_1^-(r_*) &= -i\int_{r_*}^{z(0)} e^{-iz(r_*-y)} f_1(y)dy - ie^{-iz(r_*-z(0))}\int_{-\infty}^{z(0)} e^{-izy} f_3(y)dy, \\
u_4^-(r_*) &= -i\int_{r_*}^{z(0)} e^{-iz(r_*-y)} f_4(y)dy + ie^{-iz(r_*-z(0))}\int_{-\infty}^{z(0)} e^{-izy} f_2(y)dy.
\end{aligned}$$

Hence for $\Im z > 0$ and $r_* \in \mathbb{R}$, we obtain that

$$\left((D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1}f - (D_{0,\mathbb{R}} - z)^{-1}f\right)(r_*) = \left(u^- + u^+\right)(r_*) - u(r_*), \tag{98}$$

where

$$\left(u^- + u^+\right)(r_*) - u(r_*) = \begin{pmatrix} -i\mathbf{1}_{]-\infty,z(0)]}(r_*)e^{-iz(r_*-z(0))}\int_{-\infty}^{z(0)} e^{-izy} f_3(y)dy \\ -i\mathbf{1}_{[z(0),+\infty[}(r_*)e^{iz(r_*-z(0))}\int_{z(0)}^{+\infty} e^{izy} f_4(y)dy \\ i\mathbf{1}_{[z(0),+\infty[}(r_*)e^{iz(r_*-z(0))}\int_{z(0)}^{+\infty} e^{izy} f_1(y)dy \\ i\mathbf{1}_{]-\infty,z(0)]}(r_*)e^{-iz(r_*-z(0))}\int_{-\infty}^{z(0)} e^{-izy} f_2(y)dy \end{pmatrix}. \tag{99}$$

Moreover since $\Im z > 0$, by the Cauchy-Schwartz inequality we deduce that

$$j = 1,4 \Rightarrow \left|\int_{-\infty}^{z(0)} e^{-izy} f_j(y)dy\right| \leq \frac{C_j}{\Im z}\|f_j\|, \quad j = 2,3 \Rightarrow \left|\int_{z(0)}^{+\infty} e^{izy} f_j(y)dy\right| \leq \frac{C_j}{\Im z}\|f_j\|, \tag{100}$$

with $C_j > 0$. Therefore, with (98) and (99) we obtain that for $\Im z > 0$

$$\left\|(D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1} - (D_{0,\mathbb{R}} - z)^{-1}\right\| \leq \frac{C_5}{(\Im z)^2}, \quad C_5 > 0. \tag{101}$$

Obviously, we can prove the same estimate for $\Im z < 0$ in the same way. We remark that for $\Im z \neq 0$

$$\begin{aligned} &\left\|(D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1} - (D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1}\right\| \\ &\quad = \left\|(D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1}\varsigma A_\nu (D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1}\right\| \leq \frac{C_6}{(\Im z)^2}, \end{aligned} \tag{102}$$

with $C_6 > 0$ and

$$\left\|(D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1} - (D_{0,\mathbb{R}} - z)^{-1}\right\| = \left\|(D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1}\varsigma A_\nu (D_{0,\mathbb{R}} - z)^{-1}\right\| \leq \frac{C_7}{(\Im z)^2}, \quad C_7 > 0, \tag{103}$$

since ςA_ν is bounded and $\|(D - z)^{-1}\| \leq C|\Im z|^{-1}$, $C > 0$ with D self-adjoint on $L^2_{\mathbb{R}}$. Therefore, we obtain the result by using (101), (102), (103) and the following equality :

$$\begin{aligned} &(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1} - (D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1} \\ &\quad = (D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1} - (D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1} \\ &\quad + (D_{0,]-\infty,z(0)]} \oplus D_{0,[z(0),+\infty[} - z)^{-1} - (D_{0,\mathbb{R}} - z)^{-1} \\ &\quad + (D_{0,\mathbb{R}} - z)^{-1} - (D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1}. \end{aligned}$$

■

Lemma 4.4
For $\varsigma < 0$ $(\Lambda = 0)$ and $\nu \neq (2k+1)\pi$, $k \in \mathbb{R}$, the following operators are compact in L^2_0:

$$\mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}) - \mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,\mathbb{R}}) \tag{104}$$

$$\mathbf{1}_{]-\infty,0]}(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}) - \mathbf{1}_{]-\infty,0]}(D_{\varsigma A_\nu,\mathbb{R}}) \tag{105}$$

$$\mathcal{K}^{ms}_{1,\sigma}(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}) - \mathcal{K}^{ms}_{1,\sigma}(D_{\varsigma A_\nu,\mathbb{R}}) \tag{106}$$

Proof:
To prove the result, we use the Helffer-Sjöstrand formula : given $f \in C^\infty(\mathbb{R})$ such that

$$\left|\partial_s^k f(s)\right| \leq C_k < s >^{-k}, \quad k \geq 0, \quad < s >:= \sqrt{1 + s^2}, \tag{107}$$

then there exists $\widetilde{f} \in C^\infty(\mathbb{C})$ with $\widetilde{f}_{|\mathbb{R}} = f$ and

$$\left|\partial_{\bar{z}} \widetilde{f}(z)\right| \leq C_N < \Re z >^{-N-1} |\Im z|^N, \quad C_N > 0, \tag{108}$$

$$supp \widetilde{f} \subset \{z,\ |\Im z| \leq C < \Re z >\} \tag{109}$$

such that

$$f(x) = \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\widetilde{f}(z)(x-z)^{-1}dz\wedge d\bar{z}. \tag{110}$$

Following [2], we can prove for $\varsigma < 0$ $(\Lambda = 0)$ and $\nu \neq (2k+1)\pi,\ k\in\mathbb{R}$, that

$$\left\| D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} f \right\| \geq \varsigma\nu \left\| f \right\|, \quad f\in \mathcal{D}(D_{\varsigma A_\nu,]-\infty,z(0)]}) \oplus \mathcal{D}(D_{\varsigma A_\nu,[z(0),+\infty[}).$$

Therefore, if we choose $\chi\in C^\infty(\mathbb{R})$ such that

$$\varsigma\nu \leq t \Longrightarrow \chi(t) = 1, \quad 0 \geq t \Longrightarrow \chi(t) = 0,$$

we obtain that

$$\begin{aligned} &\mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}) = \chi(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}),\\ &\mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,\mathbb{R}}) = \chi(D_{\varsigma A_\nu,\mathbb{R}}). \end{aligned}$$

The function χ satisfies property (107). By using formula (110) with the spectral theorem, we have:

$$\begin{aligned} &\chi(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}) - \chi(D_{\varsigma A_\nu,\mathbb{R}})\\ &\quad = \frac{i}{2\pi}\int_{\mathbb{C}} \partial_{\bar{z}}\widetilde{\chi}(z)\left[(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1} - (D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1}\right]dz\wedge d\bar{z}. \end{aligned} \tag{111}$$

According to the estimate (108) with $N = 2$, to prove the compactness of (104) it suffices to check that:

$$\left\|(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1} - (D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1}\right\| \leq C|\Im z|^{-2}, \quad z \in\subset \mathbb{C}\setminus\mathbb{R},$$

to obtain the norm operator convergence of (111), and the compacity in $L^2_{\mathbb{R}}$ of

$$(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[} - z)^{-1} - (D_{\varsigma A_\nu,\mathbb{R}} - z)^{-1}, \quad z\in\mathbb{C}\setminus\mathbb{R}.$$

The first property is obvious by lemma 4.3 et the second is satisfied since the previous operator is of finite rank. The result for (105) and (106) is obtained in the same way, since for the last operators the function $\mathcal{K}^{ms}_{1,\sigma}\in C^\infty(\mathbb{R})$ satisfies property (107). ■

We define V_∞ thanks to V such that

$$V_\infty := \delta I_{\mathbb{R}^4} + \varsigma A_\nu = \lim_{r_*\to+\infty} V(r_*), \quad \delta = \frac{qQ}{r_0},\ \varsigma = -m\sqrt{F(r_+)}, \tag{112}$$

where A_ν as in (68).

Lemma 4.5
Given ${}^t(0,g_2,g_3,0)\in C_0^\infty(\mathbb{R})^4$ *and* $\Lambda\geq 0$. *Then*

$$\begin{aligned} &\lim_{T\to+\infty} < (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0})[G^T]_L, [G^T]_L >_{L^2_0}\\ &\qquad\qquad = \lim_{T\to+\infty} < (\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}}) - 1)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}}, \end{aligned} \tag{113}$$

$$\begin{aligned} &\lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0})\mathbf{1}_{]-\infty,\delta]}(D_{V_\infty,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}\\ &\qquad\qquad = \lim_{T\to+\infty} < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{]-\infty,0]}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}}, \end{aligned} \tag{114}$$

Proof:
If $\varsigma = 0$ $(\Lambda > 0)$, then we have clearly

$$\begin{aligned}< (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0})[G^T]_L, [G^T]_L >_{L^2_0} \\ =< (\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}}) - 1)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}}\end{aligned} \tag{115}$$

and

$$\begin{aligned}< \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0})\mathbf{1}_{]-\infty,\delta]}(D_{V_\infty,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0} \\ =< \mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{]-\infty,0]}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}} .\end{aligned} \tag{116}$$

Now, we treat the case of $\varsigma < 0$ $(\Lambda = 0)$ for the first limit. The proof for the second is obtained by the same way. By supposing that $supp(g) \subset [0, R]$, $R > 0$ fixed, and $T > -\frac{1}{2\kappa_0}\ln(-z(0)) + \frac{1}{2\kappa_0}\ln(C_{\kappa_0}) + \frac{1}{2}$, we have $supp\left(G^T\right) \subset]z(0), 0[$. Hence

$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)\left[G^T\right]_L = 0 \oplus \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right)\left[G^T\right]_L , \tag{117}$$

with

$$\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right) = \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,0}\right) = \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right) \tag{118}$$

and

$$\mathcal{K}^{ms}_{1,\sigma_0}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)\left[G^T\right]_L = 0 \oplus \mathcal{K}^{ms}_{1,\sigma_0}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right)\left[G^T\right]_L , \tag{119}$$

with

$$\mathcal{K}^{ms}_{1,\sigma_0}\left(D_{V_\infty,0}\right) = \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,0}\right) = \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right) . \tag{120}$$

From lemma 4.4, the following operator is compact in $L^2_{\mathbb{R}}$:

$$\begin{aligned}\mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right)\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty,z(0)]} \oplus D_{\varsigma A_\nu,[z(0),+\infty[}\right) \\ - \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,\mathbb{R}}\right)\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right)\end{aligned}$$

By lemma VI.6 in [2]: $[G^T]_L \rightharpoonup 0$, $T \to +\infty$ in $L^2_{\mathbb{R}}$. Hence, we have the following limits:

$$\begin{aligned}\Big\| 0 \oplus \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right)\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,[z(0),+\infty[}\right)\left[G^T\right]_L \\ - \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(D_{\varsigma A_\nu,\mathbb{R}}\right)\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right)\left[G^T\right]_L\Big\| \to 0, \quad T \to +\infty.\end{aligned} \tag{121}$$

and

$$\begin{aligned}\lim_{T\to+\infty} < (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0})[G^T]_L, [G^T]_L >_{L^2_0} \\ = \lim_{T\to+\infty} < (\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{\varsigma A_\nu,\mathbb{R}}) - 1)\mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}} .\end{aligned} \tag{122}$$

First, we remark that using the Fourier transform $\mathcal{F}$:

$$\mathcal{F}\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right) = \left[\frac{1}{2} + \frac{1}{2\sqrt{\xi^2+\varsigma^2}}\left(i\xi\Gamma^1 + \varsigma A_\nu\right)\right]\mathcal{F}.$$

Moreover

$$\begin{aligned}
&\left\|\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{\varsigma A_\nu,\mathbb{R}})\mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,\mathbb{R}})[G^T]_L - \mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L\right\| && (123)\\
&\quad\le C_1\int_{\mathbb{R}}\left|\frac{i\xi}{|\xi|}\Gamma^1 - \frac{1}{\sqrt{\xi^2+\varsigma^2}}\left(i\xi\Gamma^1+\varsigma A_\nu\right)\right|^2\left|\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 d\xi, \quad C_1>0,\\
&\qquad + C_2\int_0^{+\infty}\left|\mathcal{K}^{ms}_{\mu_0,\sigma_0}(i\xi\Gamma^1+\varsigma A_\nu) - \mathcal{K}^{ms}_{\mu_0,\sigma_0}(i\xi\Gamma^1)\right|^2\left|\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 d\xi, \quad C_2>0,\\
&\quad= C_1\int_{\mathbb{R}}\left|\frac{i\xi}{|\xi|}\Gamma^1 - \frac{1}{\sqrt{\xi^2+B^2(T)\varsigma^2}}\left(i\xi\Gamma^1+B(T)\varsigma A_\nu\right)\right|^2|\theta(\eta)|^2 d\eta,\\
&\qquad + C_2\int_0^{+\infty}\left|\mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(i\frac{\eta}{B(T)}\Gamma^1+\varsigma A_\nu\right) - \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(i\frac{\eta}{B(T)}\Gamma^1\right)\right|^2|\theta(\eta)|^2 d\eta,\\
&\quad= I_1 + I_2 && (124)
\end{aligned}$$

By a tedious but straightforward calculations, we obtain that

$$\mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(i\frac{\eta}{B(T)}\Gamma^1+\varsigma A_\nu\right) - \mathcal{K}^{ms}_{\mu_0,\sigma_0}\left(i\frac{\eta}{B(T)}\Gamma^1\right)\longrightarrow 0, \quad T\longrightarrow+\infty, \quad \eta\ge 0. \tag{125}$$

Then, thanks to Lebesgue's theorem $\lim_{T\to+\infty} I_1 = \lim_{T\to+\infty} I_2 = 0$. We deduce that

$$\begin{aligned}
&\lim_{T\to+\infty} < (\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{\varsigma A_\nu,\mathbb{R}})-1)\mathbf{1}_{[0,+\infty[}(D_{\varsigma A_\nu,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}}\\
&\qquad\qquad = \lim_{T\to+\infty} < (\mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{0,\mathbb{R}})-1)\mathbf{1}_{[0,+\infty[}(D_{0,\mathbb{R}})[G^T]_L, [G^T]_L >_{L^2_{\mathbb{R}}}. && (126)
\end{aligned}$$

which entails the result. ■

Lemma 4.6
Given $f\in C_0^\infty(\mathbb{R})^4$ *and*

$$g(t) := \left(W^-_{0,\mathbb{R}} f\right)(1-2t), \tag{127}$$

then

$$\left\|\mathcal{J}U_V(0,T)f - [G^{T/2}]_L\right\|_0 \to 0, \quad T\to+\infty, \tag{128}$$

and

$$\mathcal{J}U_V(0,T)f \rightharpoonup 0, \quad T\to+\infty \quad in\ L^2_0. \tag{129}$$

Proof:
This result is a consequence of lemmas 6.5, 6.7 and 6.9 of [14] ■

With this previous lemma and since all operators are uniformly bounded in L^2_0 norm and $C_0^\infty(\mathbb{R})^4$ is dense in $L^2_{\mathbb{R}}$, we obtain easily:

Lemma 4.7
Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geq 0$:

$$\lim_{T\to+\infty} < (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}$$
$$= \lim_{T\to+\infty} < (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0})[G^{T/2}]_L, [G^{T/2}]_L >_{L^2_0}, \quad (130)$$
$$\lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0})\mathbf{1}_{]-\infty,\delta]}(D_{V_\infty,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}$$
$$= \lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0})\mathbf{1}_{]-\infty,\delta]}(D_{V_\infty,0})[G^{T/2}]_L, [G^{T/2}]_L >_{L^2_0}, \quad (131)$$

Lemma 4.8
The following operators are compact in L^2_0:

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0}) - \mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0}) \quad (132)$$
$$\mathbf{1}_{]-\infty,\delta]}(D_{V,0}) - \mathbf{1}_{]-\infty,\delta]}(D_{V_\infty,0}) \quad (133)$$
$$\mathcal{K}^{ms}_{1,\sigma}(D_{V,0}) - \mathcal{K}^{ms}_{1,\sigma}(D_{V_\infty,0}) \quad (134)$$

Proof:
From lemma III-10 in [2], we have the result for (132) and (133). For the last operator and as for the proof of lemma 4.4, we use the Helffer-Sjöstrand formula. We must check that:

$$\left|(D_{V,0} - z)^{-1} - (D_{V_\infty,0} - z)^{-1}\right| \leq C|\Im z|^{-2}, \quad z \in \mathbb{C} \setminus \mathbb{R} \quad (135)$$

and

$$(D_{V,0} - z)^{-1} - (D_{V_\infty,0} - z)^{-1} \quad \textit{compact in } L^2_0 \textit{ for } z \in \mathbb{C} \setminus \mathbb{R}.$$

For the second property, we remark that

$$(D_{V,0} - z)^{-1} - (D_{V_\infty,0} - z)^{-1} = (D_{V,0} - z)^{-1}(V_\infty - V)(D_{V_\infty,0} - z)^{-1} \textit{ for } z \in \mathbb{C} \setminus \mathbb{R}. \quad (136)$$

Moreover, $\lim_{r_*\to+\infty}(V_\infty(r_*) - V(r_*)) = 0$ and $(V_\infty - V) \in C^0(\mathbb{R})$. By the Sobolev embedding, we obtain that $\mathbf{1}_{[z(0),n]}(V_\infty - V)(D_{V_\infty,0} - z)^{-1}$ is compact in L^2_0 for all $n \in \mathbb{N}$ and $z \in \mathbb{C}\setminus\mathbb{R}$. As we have clearly

$$\left\|\mathbf{1}_{[z(0),n]}(V_\infty - V)(D_{V_\infty,0} - z)^{-1} - (V_\infty - V)(D_{V_\infty,0} - z)^{-1}\right\|_0 \to 0, \quad n \to +\infty,$$

we conclude that (136) is compact in L^2_0. Finally, since $(V_\infty - V) \in L^\infty(\mathbb{R})$ and $\|(D - z)^{-1}\| \leq C|\Im z|^{-1}$, $C > 0$ with D self-adjoint on L^2_0, by (136), estimate (135) is satisfied. ■

Lemma 4.9
Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geq 0$:

$$\lim_{T\to+\infty} < (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}$$
$$= \lim_{T\to+\infty} < (\mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0}) - 1)\mathbf{1}_{[\delta,+\infty[}(D_{V_\infty,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0} = 0, \quad (137)$$
$$\lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{]-\infty,\delta]}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}$$
$$= \lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_\infty,0})\mathbf{1}_{]-\infty,\delta]}(D_{V_\infty,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0} = 0. \quad (138)$$

Proof:
For $K = \mathcal{K}^{ms}_{1,\sigma_0} - 1$ and $\mathbf{1}_{\pm} = \mathbf{1}_{[\delta,+\infty[}$ or $K = \mathcal{K}^{ms}_{1,\sigma_0}$ and $\mathbf{1}_{\pm} = \mathbf{1}_{]-\infty,\delta]}$, we have:

$$\begin{aligned}K(D_{V,0})\mathbf{1}_{\pm}(D_{V,0}) =& K(D_{V,0})\left(\mathbf{1}_{\pm}(D_{V,0}) - \mathbf{1}_{\pm}(D_{V_\infty,0})\right) + \left(K(D_{V,0}) - K(D_{V_\infty,0})\right)\mathbf{1}_{\pm}(D_{V_\infty,0})\\ &+ K(D_{V_\infty,0})\mathbf{1}_{\pm}(D_{V_\infty,0}).\end{aligned}$$

We obtain the equality of the limits, by using the previous formula, lemma 4.8 and the property (129). Finally, we conclude the proof of this lemma thanks to lemmas 4.7, 4.5 and 4.2. ■

Lemma 4.10
Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geq 0$:

$$\lim_{T\to+\infty} \left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\right\|_0^2 = < W^-_{0,\mathbb{R}} f, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}} \left(1 + e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} W^-_{0,\mathbb{R}} f >_{L^2_{\mathbb{R}}}, \qquad (139)$$

with

$$\delta = \frac{qQ}{r_0}.$$

Proof:
See lemma 6.10 in [14]. ■

Proposition 4.3
Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geq 0$:

$$\lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0} = < \mathcal{K}^{ms}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}} f, W^-_{0,\mathbb{R}} f >_{L^2_{\mathbb{R}}}, \qquad (140)$$

with

$$\sigma = \frac{2\pi}{\kappa_0}.$$

Proof:
By a straightforward calculation, we have

$$\begin{aligned}&< \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}\\
&= < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}, \quad \delta := \frac{qQ}{r_0}\\
&\quad + < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{]-\infty,\delta]}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}\\
&= < \left(\mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0}) - 1\right)\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0},\\
&\quad + \left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\right\|_0^2,\\
&\quad + < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathbf{1}_{]-\infty,\delta]}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0}\end{aligned}$$

The first and the third term are treated by lemma 4.9 and the second term by lemma 4.10. ■

Proposition 4.4
Given $f \in L^2_{\mathbb{R}}$, then for $\Lambda \geq 0$:

$$\begin{aligned}\lim_{T\to+\infty} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})U_V(0,T)f, U_V(0,T)f >_{L^2_0} =& < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})W^-_{V,[z(0),+\infty[} f, W^-_{V,[z(0),+\infty[} f >_{L^2_0}\\ &+ < \mathcal{K}^{ms}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}} f, W^-_{0,\mathbb{R}} f >_{L^2_{\mathbb{R}}}, \qquad (141)\end{aligned}$$

with

$$\sigma = \frac{2\pi}{\kappa_0}.$$

Proof:
With a simple calculation, we obtain that

$$\begin{aligned}
&< \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})U_V(0,T)f, U_V(0,T)f >_{L^2_0} \\
&\quad =< \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})\mathcal{J}U_V(0,T)f, \mathcal{J}U_V(0,T)f >_{L^2_0} \\
&\quad\quad + < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V,0})(1-\mathcal{J})U_V(0,T)f, (1-\mathcal{J})U_V(0,T)f >_{L^2_0} \\
&\quad\quad + 2\Re < \mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T)f, \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f >_{L^2_0}.
\end{aligned}$$

The last term vanishes as $T \to +\infty$ thanks to limit (129) and lemma 4.1. By lemmas 4.10 and 4.1, we conclude that the two first terms are zero as $T \to +\infty$. ■

Proof of theorem 4.1 :
By lemma 4.1, the wave operator $W^-_{V_{l,\nu},[z(0),+\infty[}$ exists and is an isometry from $L^2_{\mathbb{R}}$ onto $P_{ac}(D_{V,[z(0),+\infty[})L^2_0$. Hence by using the operators (65), (63), we deduce that

$$\boldsymbol{W}^-_+ := \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}, \quad \Lambda \geq 0 \tag{142}$$

exists and is an isometry from $\boldsymbol{L}^2_{\text{BH}}$ onto $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$. By definition, we have

$$\boldsymbol{\Omega}^-_{\Lambda,\to} := \left(\boldsymbol{W}^-_{\Lambda,\to}\right)^*, \quad \Lambda \geq 0.$$

According to the chain rule theorem, the following wave operator

$$\boldsymbol{W}^-_{\Lambda,D} := \boldsymbol{\Omega}^-_{\Lambda,\to}\left(\boldsymbol{W}^-_+\right)^* : \; P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0 \to \boldsymbol{L}^2_{\Lambda,\to}, \quad \Lambda \geq 0. \tag{143}$$

is an isometry from $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$ onto $\boldsymbol{L}^2_{\Lambda,\to}$. With the help of Lebesgue theorem, proposition 4.4, the properties of the operators (65), (63) and the properties (62), (66) and (74), we obtain the following limit:

$$\begin{aligned}
&\lim_{T\to+\infty} < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\, \boldsymbol{U}(0,T)f, \boldsymbol{U}(0,T)f >_{\mathfrak{H}} \\
&\quad = \lim_{T\to+\infty}\sum_{(l,n)\in\mathcal{I}} < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(D_{V_{l,\nu},0} - \delta)U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f, U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f >_{L^2_0}, \\
&\quad = \sum_{(l,n)\in\mathcal{I}} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_{l,\nu},0})U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f, U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f >_{L^2_0}, \\
&\quad = \sum_{(l,n)\in\mathcal{I}} < \mathcal{K}^{ms}_{1,\sigma_0}(D_{V_{l,\nu},0})W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}f, W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}f >_{L^2_0} \\
&\quad\quad\quad + \sum_{(l,n)\in\mathcal{I}} < \mathcal{K}^{ms}_{1,\sigma}(D_{0,\mathbb{R}})W^-_{0,\mathbb{R}}\mathcal{R}^\nu_{ln}f, W^-_{0,\mathbb{R}}\mathcal{R}^\nu_{ln}f >_{L^2_{\mathbb{R}}}, \\
&\quad =: S_1 + S_2.
\end{aligned}$$

From the definition of $\boldsymbol{W}^{-}_{\Lambda,D}$ and $\boldsymbol{W}^{-}_{+}$, and the intertwining properties, we deduce that for $\Lambda \geq 0$

$$\begin{aligned}
S_1 &= \sum_{(l,n)\in\mathcal{I}} < W^{-}_{V_{l,\nu},[z(0),+\infty[}\mathcal{K}^{ms}_{1,\sigma_0}(D_{V,\mathbb{R}})\mathcal{R}^{\nu}_{ln}f, W^{-}_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^{\nu}_{ln}f >_{L^2_0},\\
&=< \boldsymbol{W}^{-}_{+}\mathcal{K}^{ms}_{1,\sigma_0}(\boldsymbol{D}_{\mathrm{BH}}+\delta)f, \boldsymbol{W}^{-}_{+}f >_{\boldsymbol{L}^2_0}\\
&=< \boldsymbol{W}^{-}_{\Lambda,D}\,\boldsymbol{W}^{-}_{+}\mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\mathrm{BH}})f, \boldsymbol{W}^{-}_{\Lambda,D}\,\boldsymbol{W}^{-}_{+}f >_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}\\
&=< \boldsymbol{\Omega}^{-}_{\Lambda,\rightarrow}\,\mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\mathrm{BH}})f, \boldsymbol{\Omega}^{-}_{\Lambda,\rightarrow}\,f >_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}\\
&=< \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\rightarrow})\boldsymbol{\Omega}^{-}_{\Lambda,\rightarrow}\,f, \boldsymbol{\Omega}^{-}_{\Lambda,\rightarrow}\,f >_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}.
\end{aligned}$$

We define

$$\boldsymbol{\Omega}^{-}_{\leftarrow} := \left(\boldsymbol{W}^{-}_{\leftarrow}\right)^*,$$

and remark that

$$\mathcal{P}_r\boldsymbol{D}_{\leftarrow}\,\mathcal{P}_r^{-1} = \bigoplus_{(l,n)\in\mathcal{I}}\mathcal{E}^{\nu}_{ln}D_{0,\mathbb{R}}\,\mathcal{R}^{\nu}_{ln} - \delta, \quad \delta = \frac{qQ}{r_0}.$$

Hence, with (62) and (82), we have

$$\begin{aligned}
S_2 &=< \mathcal{P}_r\mathcal{K}^{ms}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}^{-}_{\leftarrow}f, \mathcal{P}_r\boldsymbol{\Omega}^{-}_{\leftarrow}f >_{\boldsymbol{L}^2_{\mathrm{BH}}}, \quad \boldsymbol{L}^2_{\mathrm{BH}} = \mathcal{P}_r\boldsymbol{L}^2_{\leftarrow},\\
&=< \mathcal{K}^{ms}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}^{-}_{\leftarrow}f, \boldsymbol{\Omega}^{-}_{\leftarrow}f >_{\boldsymbol{L}^2_{\leftarrow}}, \quad \mu = e^{\sigma\delta}, \quad \sigma := \frac{2\pi}{\kappa_0}, \quad \delta := \frac{qQ}{r_0}.
\end{aligned}$$

Therefore, we obtain limit (55). ■

4.3 Proof of theorem 3.1

By the identity of polarization, it is sufficient to evaluate for $\Phi \in C_0^\infty(\mathcal{M}_{\mathrm{coll}})^4$ the following limit:

$$\lim_{T\to+\infty}\omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi^T)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi^T)).$$

Since for $T > 0$ large enough, we have:

$$S_{\mathrm{coll}}\Phi^T = \boldsymbol{U}(0,T)S_{\mathrm{bh}}\Phi, \quad S_{\mathrm{bh}}\Phi := \int_{\mathbb{R}}\boldsymbol{U}(-t)\Phi(t)dt,$$

we obtain that

$$\begin{aligned}
\lim_{T\to+\infty}\omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi^T)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi^T)) &= \lim_{T\to+\infty} < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)S_{\mathrm{coll}}\Phi^T, S_{\mathrm{coll}}\Phi^T >_{\mathfrak{H}},\\
&= \lim_{T\to+\infty} < \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)S_{\mathrm{bh}}\Phi, \boldsymbol{U}(0,T)S_{\mathrm{bh}}\Phi >_{\mathfrak{H}}. \qquad (144)
\end{aligned}$$

Therefore, thanks to limit (144) of theorem 4.1, we deduce that for $\Lambda \geq 0$:

$$\begin{aligned}\lim_{T\to+\infty} \omega_{\mathcal{M}_{\mathrm{coll}}}(\boldsymbol{\Psi}^*_{\mathrm{coll}}(\Phi^T)\boldsymbol{\Psi}_{\mathrm{coll}}(\Phi^T)) &=< \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\rightarrow})\boldsymbol{\Omega}^-_{\Lambda,\rightarrow} S_{\mathrm{bh}}\Phi, \boldsymbol{\Omega}^-_{\Lambda,\rightarrow} S_{\mathrm{bh}}\Phi >_{\boldsymbol{L}^2_{\Lambda,\rightarrow}} \\ &\quad + < \mathcal{K}^{ms}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})\boldsymbol{\Omega}^-_{\leftarrow} S_{\mathrm{bh}}\Phi, \boldsymbol{\Omega}^-_{\leftarrow} S_{\mathrm{bh}}\Phi >_{\boldsymbol{L}^2_{\leftarrow}} \\ &=< \mathcal{K}^{ms}_{\mu_0,\sigma_0}(\boldsymbol{D}_{\Lambda,\rightarrow})S_{\Lambda,\rightarrow}\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi, S_{\Lambda,\rightarrow}\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi >_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \\ &\quad + < \mathcal{K}^{ms}_{\mu,\sigma}(\boldsymbol{D}_{\leftarrow})S_{\leftarrow}\boldsymbol{\Omega}^-_{\leftarrow}\Phi, S_{\leftarrow}\boldsymbol{\Omega}^-_{\leftarrow}\Phi >_{\boldsymbol{L}^2_{\leftarrow}} \\ &= \omega^{\delta,\sigma}_{\mathrm{Haw}}(\boldsymbol{\Psi}^*_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi)\boldsymbol{\Psi}_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi)) \\ &\qquad + \omega^{\delta_0,\sigma_0}_{\mathrm{KMS}}(\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi)).\end{aligned}$$

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