The Hawking Effect for Spin 1/2 Fields

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Abstract. - We prove the Hawking effect, for a gravitational collapse of charged star in the case of a charged massive Dirac field.

Key Words. - Hawking effect, Quantum field, Gravitational collapse, Scattering, Black-hole, Dirac equation, Reissner-Nordstrøm metric, General relativity.

1 Introduction.

In this paper, we investigate the Hawking effect [14] in the case of the Dirac quantum field. We adopt the semi-classical approximation by supposing that the space-time curvature influences the fields, but the back-reaction on the metric is neglected. Then, we prove the emergence of a thermal state at the last moments of a gravitational collapse which is interpreted by a static observer at infinity as an outgoing flux of particles and anti-particles. Moreover, the black-hole preferentially emits massive spin 1/2 particles whose charge is of same sign as its own charge.

The Hawking effect and more generally the quantum effects in the vicinity of a black-hole have been the subject of numerous studies, we mention only the works that we have used: [5], [11], [25], [26].

A first mathematical study of the Hawking radiation was undertaken by J. Dimock and B. S. Kay [10]. In this work the authors consider the case of a Schwarzschild black hole for a Klein-Gordon field. By quantizing suitably this field in the vicinity of the past horizon of the black hole, the authors show that an observer located at infinity future observes the Hawking radiation. The case that was initially considered by S. Hawking of gravitational collapse in the Fock vacuum was examined by A. Bachelot. In a first time and for a field of Klein-Gordon [1], the author showed that a plunging observer in the future Schwarzschild black hole observes the Hawking radiation when he crosses the horizon of the black hole. In a second paper and for the same field, A. Bachelot obtained the proof of the Hawking effect [3]: a fixed observer in Schwarzschild variables observes at last moments of collapse in his own proper time, an outgoing Hawking thermal flux coming from the horizon of the future Schwarzschild black hole. In [4], this same author extends his study [1] to the case of charged Dirac field for a plunging observer in a charged black hole resulting from a gravitational collapse.

Just like that was done for the field of Klein-Gordon in [3], our contribution to this program of study is to prove the Hawking effect for charged Dirac field of the point of view of a fixed observer in Schwarzschild variables for a collapsing charged star. More precisely, in this work (and as for those of A. Bachelot) we consider a very simplified model of gravitational collapse, for which the star is modelled by a reflecting sphere: the properties of the star surface are given by the boundary condition for the Dirac field on this surface. Here, we chose the *MIT bag* boundary condition [6] which is conservative and which causes a reflexion of the fields on the star surface like occurs for a bosonic field by using a Dirichlet condition. These simplifying assumptions enable us to avoid difficult studies of the interactions between the fields and the fluid which composes star and of the behavior of this fluid at the time of gravitational collapse via the Einstein-Maxwell equations. Moreover, we suppose that the spherical symmetry of the charged star is preserved during the collapse, hence, outside this one and by the Birkhoff theorem, the DeSitter-Reissner-Nordstrøm or the Reissner-Nordstrøm spaces time are relevants. The gravitational collapse occurs in the Fock vacuum. Although this last assumption is not physically correct in the case of DeSitter-Reissner-Nordstrøm space time (see [13]), the mathematical proof remains valid. Indeed, in this case, it would be preferable to consider a thermal state whose temperature is that of Gibbons-Hawking associated to the cosmological horizon. A forthcoming work will be to study the Hawking effect for Dirac field in (DeSitter-)Reissner-Nordstrøm space time by considering the gravitational collapse in a thermal bath of arbitrary temperature.

This article is organized as follows: In the second part, we define the geometrical framework for a charged collapsing star described by the globally hyperbolic manifold $(\mathcal{M}_{\rm coll}, g)$. This collapse creates the (DeSitter-)Reissner-Nordstrøm space-time $(\mathcal{M}_{\rm bh}, g)$ produced by a charged black-hole. In the third part, we define the Dirac equation for massive charged spin 1/2 field on $(\mathcal{M}_{\rm coll}, g)$ with *MIT bag* boundary conditions on the star surface. The mixed problem is well-posed. In the fourth part, we study the scattering theory for the massive charged Dirac field in the charged eternal black-hole $(\mathcal{M}_{\rm bh}, g)$. To do this, we introduce the useful wave operators at the horizon and at infinity. More particularly, we extend the studies of [16], [21] and [18, 19], in proving the asymptotic completeness for the classical wave operators at the horizon and infinity when we consider the curved DeSitter-Reissner-Nordstrøm space-time. In the fifth part, we construct the local algebra of observable $\mathfrak{U}(\mathcal{M}_{\rm coll})$ as in [8] and [9], using the Dirac-Fermi Fock representation on some particular Cauchy hyper-surface. We define the KMS-state involving the (Hawking) temperature and the chemical potential. In this same section we state the main theorem of this work using the mathematical objects of the previous part. We interpret the result as a thermal state given by a KMS-state which is independent on the behavior of the collapse and boundary condition on the star for the Dirac field. The last section is devoted to the proofs of the technical results useful to demonstrate the main theorem of this article.

2 Geometrical description of a gravitational collapse.

We introduce the general geometrical framework describing the creation of a black-hole by an idealized star collapsing. First, we consider the (DeSitter-)Reissner-Nordstrøm space-time outside a charged, static eternal black-hole in an expanding universe, as the globally hyperbolic manifold $(\mathcal{M}_{\rm bh}, g)$,

$$\begin{aligned}
&\mathcal{M}_{\rm bh} = \mathbb{R}_t \times]r_0, r_+[\times S^2_\omega, \quad 0 < r_0 < r_+ \leq +\infty,\\
&g_{ab}dx^a dx^b = F(r)dt^2 - F^{-1}(r)dr^2 - r^2 d\omega^2,\\
&d\omega^2 = d\theta^2 + sin^2\theta d\varphi^2,\ \omega = (\theta, \varphi) \in [0, \pi] \times [0, 2\pi[,\\
&F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}.
\end{aligned} \tag{1}$$

Here, $Q \in \mathbb{R}$, $M > 0$, $\Lambda \geq 0$, r_0 and r_+ are respectively the electric charge, the mass, the cosmological constant, the radius of the horizon of the black-hole and the radius of the cosmological horizon. We have

$$F(r_0) = F(r_+) = 0, \quad 2\kappa_0 = F^{'}(r_0) > 0, \quad 2\kappa_+ = F^{'}(r_+) < 0,\ r \in]r_0, r_+[\Rightarrow F(r) > 0.$$

with κ_0, κ_+ the surface gravity at the black hole horizon and at the cosmological horizon. If $\Lambda = 0$ then

$$\begin{aligned}
&F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad 0 < |Q| \leq M,\\
&r_0 = M + \sqrt{M^2 - Q^2}, \quad r_+ = +\infty,
\end{aligned}$$

and the globally hyperbolic manifold $(\mathcal{M}_{\rm bh}, g)$ describes the Reissner-Nordstrøm space-time which is asymptotically flat at spatial infinity. We introduce a radial coordinate r_*, which straightens the radial null geodesics:

$$r_* = \frac{1}{2\kappa_0}\left[\ln(r-r_0) - \int_{r_0}^{r}\left(\frac{1}{x-r_0} - \frac{2\kappa_0}{F(x)}\right)dx\right] + c, \quad r \in]r_0, r_+[, \quad c \in \mathbb{R}, \tag{2}$$

$$\frac{dr_*}{dr} = F^{-1}. \tag{3}$$

This coordinate shifts the horizon of the black-hole to the negative infinity and the cosmological horizon to the positive infinity.

As we consider a black-hole created by the collapse of spherical charged star, if the exact spherical symmetry of the star in collapsing is maintained, outside of it, the (DeSitter-)Reissner-Nordstrøm geometry is relevant thanks to Birkoff's theorem [15], [20]. Hence the space-time outside the spherical charged star with r_*-radius $z(t)$, $t \in \mathbb{R}$, is the manifold $(\mathcal{M}_{\text{coll}}, g)$ such that :

$$\begin{aligned}\mathcal{M}_{\text{coll}} &:= \left\{(t, r_*, \omega) \in \mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega, \quad r_* \geq z(t)\right\}, \\ &= \cup_{t\in\mathbb{R}}\left(\{t\} \times]z(t), +\infty[_{r_*} \times S^2_\omega\right).\end{aligned} \tag{4}$$

Following the general geometrical discussion about the same problem in [2] and [4], the reasonable assumptions of generic collapse lead to the following properties for $z(t)$:

$$z \in C^2(\mathbb{R}); \ \forall t \in \mathbb{R}, \ -1 < \dot{z}(t) \leq 0, \tag{5}$$

$$z(t) = -t - C_{\kappa_0} e^{-2\kappa_0 t} + \varpi(t), \ C_{\kappa_0} > 0, \ |\varpi(t)| + |\dot{\varpi}(t)| = \mathcal{O}\left(e^{-4\kappa_0 t}\right), \ t \to +\infty. \tag{6}$$

We suppose the star stationary in the past. Moreover, we arbitrarily choose c in (2), such that for all $t \leq 0$,

$$z(t) = z(0) < 0.$$

If we consider ray of light leaving x_0 at $t = 0$, with $z(0) \leq x_0 < 0$, then $\tau(x_0)$ is the time where the ray is reflected by the surface of the star,

$$\mathcal{S} := \bigcup_{t\in\mathbb{R}} \{(t, z(t))\} \times S^2_\omega,$$

such that $\tau(x_0)$ is the unique solution of

$$z(\tau(x_0)) + \tau(x_0) = x_0. \tag{7}$$

Thanks to the property (6), we have also (see [1]):

$$\tau(x_0) = -\frac{1}{2\kappa_0}\ln(-x_0) + \frac{1}{2\kappa_0}\ln(C_{\kappa_0}) + \mathcal{O}(x_0), \quad x_0 \to 0^-, \quad C_{\kappa_0} > 0, \tag{8}$$

$$1 + \dot{z}(\tau(x_0)) = -2\kappa_0 x_0 + \mathcal{O}(x_0^2), \quad x_0 \to 0^-. \tag{9}$$

3 The Dirac equation.

For the spin 1/2 particles with real charge q and mass $m > 0$, the Dirac equation on $(\mathcal{M}_{\text{coll}}, g)$, has the general form (see and [4] and [22])

$$\left[\frac{i\gamma^0}{\sqrt{F}}\left(\partial_t + i\frac{qQ}{r}\right) + \frac{i\gamma^1}{\sqrt{F}}\left(\partial_{r_*} + \frac{F}{r} + \frac{F'}{4}\right) + \frac{i\gamma^2}{r}\left(\partial_\theta + \frac{1}{2}\cot\theta\right) + \frac{i\gamma^3}{r\sin\theta}\partial_\varphi - m\right]\Psi = 0 \tag{10}$$

where the Dirac matrices γ^k, satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}\boldsymbol{I}_{\mathbb{R}^4}, \quad a, b = 0, .., 3, \quad \eta^{ab} = \text{Diag}(1, -1, -1, -1). \tag{11}$$

$$\gamma^0 = i\begin{pmatrix} 0 & \sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \quad \gamma^k = i\begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix} \quad k = 1, 2, 3, \tag{12}$$

with the Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

On the star surface, we put the following boundary condition, written for $(t, r_*, \omega) \in \mathcal{S}$, as

$$n_j \gamma^j \Psi(t, r_*, \omega) = \mathcal{B}\Psi \tag{14}$$

where n_j is the outgoing normal of subset of $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega$ and $\mathcal{B}$ some operator local in time, rotationally invariant and which conserves the L^2 norm. We choose $\mathcal{B}$ such that (14) forms a family indexed by a parameter ν of non equivalent boundary conditions: the generalized *MIT* boundary condition (see [6]), $\mathcal{B}^\nu_{\text{MIT}}$ defined by

$$\mathcal{B}^\nu_{\text{MIT}} := ie^{i\nu\gamma^5}\Psi(t, r_*, \omega), \quad \gamma^5 := -i\gamma^0\gamma^1\gamma^2\gamma^3 = diag(1, 1, -1, -1) \tag{15}$$

where the parameter ν is the chiral angle. We suppose that $\nu \in \mathbb{R}$ if $m > 0$ with $r_+ < +\infty$, and $\nu \neq (2k+1)\pi$, $k \in \mathbb{Z}$ if $m > 0$ with $r_+ = +\infty$. We introduce the Hilbert spaces:

$$\boldsymbol{L}^2_t := L^2(]z(t), +\infty[_{r_*} \times S^2_\omega, r^2 F^{1/2}(r) dr_* d\omega)^4, \quad \boldsymbol{L}^2_{\text{BH}} := L^2(\mathbb{R}_{r_*} \times S^2_\omega, r^2 F^{1/2}(r) dr_* d\omega)^4. \tag{16}$$

The norms of these spaces are denoted by $\|.\|_t$ and $\|.\|$. Moreover for $\Phi \in \boldsymbol{L}^2_t$,

$$\|\Phi\|_t = \|[\Phi]_L\|, \quad [\Phi]_L(r_*, \omega) = \begin{cases} \Phi(r_*, \omega) & r_* \in]z(t), +\infty[_{r_*} \\ 0 & r_* \in \mathbb{R} \backslash]z(t), +\infty[_{r_*} \end{cases}.$$

Hence, respectively, on $(\mathcal{M}_{\text{coll}}, g)$ and on $(\mathcal{M}_{\text{bh}}, g)$, we consider the hyperbolic mixed problems:

$$\partial_t \Psi = i\boldsymbol{D}_t \Psi, \quad z(t) < r_*, \tag{17}$$

$$\frac{\dot{z}\gamma^0 - \gamma^1}{\sqrt{1 - \dot{z}^2}} \Psi(t, z(t)) = ie^{i\nu\gamma^5}\Psi(t, z(t)) \tag{18}$$

$$\Psi(t = s, .) = \Psi_s(.) \in \boldsymbol{L}^2_s, \tag{19}$$

and

$$\partial_t \Psi = i\boldsymbol{D}_{\text{BH}} \Psi \tag{20}$$

$$\Psi(t = 0, .) = \Psi_{\text{BH}}(.) \in \boldsymbol{L}^2_{\text{BH}}, \tag{21}$$

with, $\boldsymbol{D}_t$ defined on $\boldsymbol{L}^2_t$ and $\boldsymbol{D}_{\text{BH}}$ defined on $\boldsymbol{L}^2_{\text{BH}}$, such that:

$$\boldsymbol{D}_t, \boldsymbol{D}_{\text{BH}} = -\frac{qQ}{r} + \Gamma^1 \left(\partial_{r_*} + \frac{F(r)}{r} + \frac{F'(r)}{4} \right) + \sqrt{F(r)} \left(\frac{\Gamma^2}{r}(\partial_\theta + \frac{1}{2}\cot\theta) + \frac{\Gamma^3}{r\sin\theta}\partial_\varphi + \Gamma^4 \right), \tag{22}$$

$$\Gamma^1 := i\gamma^0\gamma^1 = i\text{Diag}(-1, 1, 1, -1), \quad \Gamma^2 := i\gamma^0\gamma^2, \quad \Gamma^3 := i\gamma^0\gamma^3, \quad \Gamma^4 := -m\gamma^0, \tag{23}$$

$$\mathcal{D}(\boldsymbol{D}_t) = \left\{ \Psi \in \boldsymbol{L}^2_t, \ \boldsymbol{D}_t\Psi \in \boldsymbol{L}^2_t; \ \frac{\dot{z}\gamma^0 - \gamma^1}{\sqrt{1 - \dot{z}^2}} \Psi(z(t), \omega) = ie^{i\nu\gamma^5}\Psi(z(t), \omega) \right\} \tag{24}$$

and

$$\mathcal{D}(\boldsymbol{D}_{\text{BH}}) = \left\{ \Psi \in \boldsymbol{L}^2_{\text{BH}}, \ \boldsymbol{D}_{\text{BH}}\Psi \in \boldsymbol{L}^2_{\text{BH}} \right\}. \tag{25}$$

Proposition III.2 in [4] gives the solution $\Psi(t)$ of the hyperbolic problem (17), (18) and (19) expressed with the propagator $\boldsymbol{U}(t, s)$:

Proposition 3.1
Given $\Psi_s \in \mathcal{D}(\boldsymbol{D}_s)$, there exists $[\Psi(.)]_L = [\boldsymbol{U}(.,s)\Psi_s]_L \in C^1(\mathbb{R}_t, \boldsymbol{L}^2_{BH})$ solution of (17), (18) and (19) such that, for all $t \in \mathbb{R}$

$$\Psi(t) \in \mathcal{D}(\boldsymbol{D}_s).$$

Moreover,

$$\|\Psi(t)\|_t = \|\Psi_s\|_s$$

and $\boldsymbol{U}(t,s)$ can be extended in an isometric strongly continuous propagator from $\boldsymbol{L}^2_s$ onto $\boldsymbol{L}^2_t$.

For the eternal black-hole, we have (see theorem 4.1 in [17]):

Proposition 3.2
$\boldsymbol{D}_{BH}$ is a densely defined self-adjoint operator on $\boldsymbol{L}^2_{BH}$, hence the Cauchy problem (20) (21) has a unique solution $\Psi \in C^0(\mathbb{R}_t, \boldsymbol{L}^2_{BH})$, given by the strongly continuous unitary group $\boldsymbol{U}(t) := e^{it\boldsymbol{D}_{BH}}$:

$$\Psi(t) = \boldsymbol{U}(t)\Psi_{BH}, \quad \Psi(0) = \Psi_{BH}, \quad \|\Psi(t)\| = \|\Psi_{BH}\|.$$

4 Scattering by an eternal black-hole

Since the Hawking effect arises from an asymptotic study of the fields, we define the wave operators for the eternal charged black-hole. Near the black-hole horizon (resp. near the cosmological horizon when $\Lambda \neq 0$), we compare the solution of (20) on $\boldsymbol{L}^2_{\mathrm{BH}}$ with the solution of

$$\partial_t \Psi_{\leftarrow} = i\boldsymbol{D}_{\leftarrow}\Psi_{\leftarrow} \quad \left(resp.\ \partial_t\Psi_{\rightarrow} = \boldsymbol{D}_{\Lambda,\rightarrow}\Psi_{\rightarrow}\right)$$

where

$$\boldsymbol{D}_{\leftarrow} := \Gamma^1\partial_{r_*} - \frac{qQ}{r_0} \quad \left(resp.\ \boldsymbol{D}_{\Lambda,\rightarrow} := \Gamma^1\partial_{r_*} - \frac{qQ}{r_+}\right)$$

is self-adjoint on

$$\boldsymbol{L}^2_{\leftarrow} := L^2(\mathbb{R}_{r_*} \times S^2_\omega;\ dr_* d\omega)^4, \quad (resp.\ \boldsymbol{L}^2_{\Lambda,\rightarrow} := \boldsymbol{L}^2_{\leftarrow}, \quad \Lambda > 0),$$

with the dense domain

$$\mathcal{D}(\boldsymbol{D}_{\leftarrow}) = H^1(\mathbb{R}_{r_*}; L^2(S^2_\omega))^4 \quad \left(resp.\ \mathcal{D}(\boldsymbol{D}_{\Lambda,\rightarrow}) = H^1(\mathbb{R}_{r_*}; L^2(S^2_\omega))^4\right).$$

Thanks to the form of Γ^1, we define the subspaces of outgoing and incoming waves $\boldsymbol{L}^{2+}_{\leftarrow}$ and $\boldsymbol{L}^{2-}_{\leftarrow}$ such that $\boldsymbol{L}^2_{\leftarrow} = \boldsymbol{L}^{2+}_{\leftarrow} \oplus \boldsymbol{L}^{2-}_{\leftarrow}$,

$$\begin{aligned} &\boldsymbol{L}^{2+}_{\leftarrow} := \{\Psi \in \boldsymbol{L}^2_{\leftarrow};\ \Psi_2 = \Psi_3 = 0\}, \quad \boldsymbol{L}^{2-}_{\leftarrow} := \{\Psi \in \boldsymbol{L}^2_{\leftarrow};\ \Psi_1 = \Psi_4 = 0\}, \\ &\boldsymbol{L}^2_{\Lambda,\rightarrow} = \boldsymbol{L}^{2+}_{\Lambda,\rightarrow} \oplus \boldsymbol{L}^{2-}_{\Lambda,\rightarrow}, \quad \boldsymbol{L}^{2+}_{\Lambda,\rightarrow} := \boldsymbol{L}^{2+}_{\leftarrow}, \quad \boldsymbol{L}^{2-}_{\Lambda,\rightarrow} := \boldsymbol{L}^{2-}_{\leftarrow}. \end{aligned} \tag{26}$$

We introduce for the two asymptotic regions, respectively the identifying operator between $\boldsymbol{L}^2_{\leftarrow}$ and $\boldsymbol{L}^2_{\mathrm{BH}}$ and the one between $\boldsymbol{L}^2_{\Lambda,\rightarrow}$ and $\boldsymbol{L}^2_{\mathrm{BH}}$:

$$\begin{aligned} &\mathcal{J}_{\leftarrow} : \Psi^{\pm}(r_*,\omega) \mapsto \chi_{\leftarrow}(r_*) r^{-1} F^{-1/4}(r)\Psi^{\pm}(r_*,\omega), \quad \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, \\ &\mathcal{J}_{\Lambda,\rightarrow} : \Psi^{\pm}(r_*,\omega) \mapsto \chi_{\rightarrow}(r_*) r^{-1} F^{-1/4}(r)\Psi^{\pm}(r_*,\omega), \quad \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\Lambda,\rightarrow}, \end{aligned}$$

where $\chi_{\leftarrow}$ and $\chi_{\rightarrow}$ are cut-off functions,

$$\chi_{\leftarrow} \in C^\infty(\mathbb{R}_{r_*}), \quad \exists\, a,b \in \mathbb{R},\ 0 < a < b < 1 \quad \chi_{\leftarrow}(r_*) = \begin{cases} 1 & r_* < a \\ 0 & r_* > b \end{cases}, \quad \chi_{\rightarrow} = 1 - \chi_{\leftarrow}. \tag{27}$$

If $\Lambda > 0$, we define the wave operators $\boldsymbol{W}^{\pm}_{\leftarrow}$ at the black-hole horizon and $\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}$ at the cosmological horizon, by

$$\boldsymbol{W}^{\pm}_{\leftarrow}\Psi^{\pm} = \lim_{t\to\pm\infty} \boldsymbol{U}(-t)\mathcal{J}_{\leftarrow}\, e^{it\boldsymbol{D}_{\leftarrow}}\,\Psi^{\pm} \quad \text{in} \quad \boldsymbol{L}^2_{\text{BH}}, \quad \Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, \tag{28}$$

$$\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\,\Psi^{\mp} = \lim_{t\to\pm\infty} \boldsymbol{U}(-t)\mathcal{J}_{\Lambda,\rightarrow}\, e^{it\boldsymbol{D}_{\Lambda,\rightarrow}}\,\Psi^{\mp} \quad \text{in} \quad \boldsymbol{L}^2_{\text{BH}}, \quad \Psi^{\mp} \in \boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}. \tag{29}$$

When $\Lambda = 0$, the space-time is asymptotically flat at the infinity. Hence, we compare the solutions of (17) on $\boldsymbol{L}^2_{\text{BH}}$ with the solution $\Psi_{\rightarrow}$ of the Dirac equation on Minkowski space-time with spherical coordinates $(\rho,\omega) \in \mathbb{R}^+_* \times [0,\pi] \times [0,2\pi[$, putting $r_* = \rho > 0$ to avoid artificial long-range interactions :

$$\partial_t \Psi_{\rightarrow} = i\boldsymbol{D}_{0,\rightarrow}\,\Psi_{\rightarrow}$$

where

$$\boldsymbol{D}_{0,\rightarrow} := \Gamma^1\left(\partial_\rho + \frac{1}{\rho}\right) + \frac{\Gamma^2}{\rho}(\partial_\theta + \frac{1}{2}\cot\theta) + \frac{\Gamma^3}{\rho\sin\theta}\partial_\varphi + \Gamma^4,$$

is self-adjoint on

$$\boldsymbol{L}^2_{0,\rightarrow} := L^2(\mathbb{R}^+_\rho \times S^2_\omega;\ \rho^2 d\rho d\omega)^4$$

with the dense domain

$$\mathcal{D}(\boldsymbol{D}_{0,\rightarrow}) = H^1(\mathbb{R}^+_\rho \times S^2_\omega;\ \rho^2 d\rho d\omega)^4.$$

We define the Dirac operator with Cartesian coordinates $\bar{\boldsymbol{D}}_{0,\rightarrow}$ on $L^2(\mathbb{R}^3_x)^4$, with the help of the isometry $\mathcal{T}$ between $L^2(\mathbb{R}^3_x)^4$ and $\boldsymbol{L}^2_{0,\rightarrow}$, such that :

$$\mathcal{T} : \bar{\Psi}(x) \mapsto \Psi(\rho,\omega) = \mathcal{T}\bar{\Psi}(x), \quad \mathcal{T} = e^{\frac{\varphi}{2}\gamma^1\gamma^2} e^{-\frac{\pi}{4}\gamma^2\gamma^3} e^{\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\gamma^1\gamma^2}$$

$$\mathcal{T}\boldsymbol{D}_{0,\rightarrow}\mathcal{T}^{-1} = \bar{\boldsymbol{D}}_{0,\rightarrow} = \alpha.\boldsymbol{p} + m\beta,\ \alpha = i(\Gamma^1,\Gamma^2,\Gamma^3),\ \beta = -\gamma^0,\ \boldsymbol{p} = -i\nabla.$$

The previous comparison involves long-range perturbations due to the mass and the charge. Then, as in [17] and [19], we construct the Dollard-modified propagator $\boldsymbol{U}_{0,\rightarrow}(t)$:

$$\boldsymbol{U}_{0,\rightarrow}(t) := \mathcal{T}u(t)\mathcal{T}^{-1},\ u(t) := e^{it\lambda(\boldsymbol{p})}e^{iX^+(t)}P^0_+ + e^{-it\lambda(\boldsymbol{p})}e^{iX^-(t)}P^0_-, \tag{30}$$

$$X^{\pm}(t) := \pm m^2 M\frac{\log(t)}{|\boldsymbol{u}(\boldsymbol{p})|\lambda(\boldsymbol{p})} - qQ\frac{\log(t)}{|\boldsymbol{u}(\boldsymbol{p})|},\ \lambda(\boldsymbol{p}) := \sqrt{|\boldsymbol{p}|^2+m^2},\ \boldsymbol{u}(\boldsymbol{p}) := \boldsymbol{p}/\lambda(\boldsymbol{p}),$$

$$\log(t) := t|t|^{-1}\ln|t|,\ P^0_{\pm} := 1/2(1 \mp \bar{\boldsymbol{D}}_{0,\rightarrow}/\lambda(\boldsymbol{p})).$$

We define the bounded identifying operator $\mathcal{J}_{0,\rightarrow}$ between $\boldsymbol{L}^2_{0,\rightarrow}$ and $\boldsymbol{L}^2_{\text{BH}}$:

$$(\mathcal{J}_{0,\rightarrow}\Psi)(r_*,\omega) := \begin{cases} \chi_{\rightarrow}(\rho = r_*)r^{-1}F^{-1/4}(r)r_*\Psi(\rho = r_*,\omega) & r_* > 0 \\ 0 & r_* \leq 0 \end{cases},\ \forall\Psi \in \boldsymbol{L}^2_{0,\rightarrow},$$

and in the case of $\Lambda = 0$ the wave operator $\boldsymbol{W}^{\pm}_{0,\rightarrow}$ at infinity, for all $\Psi \in \boldsymbol{L}^2_{0,\rightarrow}$:

$$\boldsymbol{W}^{\pm}_{0,\rightarrow}\,\Psi = \lim_{t\to\pm\infty} \boldsymbol{U}(-t)\mathcal{J}_{0,\rightarrow}\boldsymbol{U}_{0,\rightarrow}(t)\Psi \quad \text{in} \quad \boldsymbol{L}^2_{\text{BH}}, \tag{31}$$

Then, we state the theorem which is proved in the last part of this work:

Theorem 4.1
The operators $\boldsymbol{W}^{\pm}_{\leftarrow}$, $\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}$ and $\boldsymbol{W}^{\pm}_{0,\rightarrow}$, respectively on $\boldsymbol{L}^{2\pm}_{\leftarrow}$, $\boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}$ and $\boldsymbol{L}^2_{0,\rightarrow}$ exist and are independent of $\chi_{\leftarrow}$, $\chi_{\rightarrow}$ and $\chi_{\rightarrow}$ satisfying (114). Moreover :

$$\|\boldsymbol{W}^{\pm}_{\leftarrow}\Psi^{\pm}\| = \|\Psi^{\pm}\|_{\boldsymbol{L}^2_{\leftarrow}}, \quad \forall\Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow}, \quad (\Lambda \geq 0, \quad m \geq 0),$$

$$\|\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\,\Psi^{\mp}\| = \|\Psi^{\mp}\|_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \quad \forall\Psi^{\mp} \in \boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}, \quad (\Lambda > 0, \quad m \geq 0),$$

$$\|\boldsymbol{W}^{\pm}_{0,\rightarrow}\,\Psi\| = \|\Psi\|_{\boldsymbol{L}^2_{0,\rightarrow}}, \quad \forall\Psi \in \boldsymbol{L}^2_{0,\rightarrow}, \quad (\Lambda = 0, \quad m > 0),$$

and

$$Ran\left(\boldsymbol{W}^{\pm}_{\leftarrow}\oplus\boldsymbol{W}^{\pm}_{\Lambda,\rightarrow}\right)=\boldsymbol{L}^2_{BH},\quad(\Lambda\geq 0).$$

5 Dirac Quantum Field and Hawking effect

5.1 Second quantization of the Dirac fields

We define the framework of the Quantum Field Theory to describe the Hawking effect. We use the approach of the algebras of local observables on curved space-time introduced by J. Dimock in [8] and [9]. First, we define the Fermi-Dirac Fock space which describes the state with an arbitrary number of non interacting charged fermions. Given, $(\mathfrak{H},<.,.>_{\mathfrak{H}})$ a complex Hilbert space and Υ the charge conjugation (see [24] section 1.4.6), then we split $\mathfrak{H}$ into two orthogonal spectral subspaces

$$\mathfrak{H}=\mathfrak{H}_+\oplus\mathfrak{H}_-,\quad \mathfrak{H}_+:=P_+\mathfrak{H},\quad \mathfrak{H}_-:=P_-\mathfrak{H},\tag{32}$$

where, P_+ and P_- are the spectral projectors on positive and negative subspaces. We define, $\mathfrak{F}^{(1)}(\mathfrak{H}_+)$ and $\mathfrak{F}^{(1)}(\mathfrak{H}_-)$, respectively the one particle space and the one anti-particle space such that

$$\mathfrak{F}^{(1)}(\mathfrak{H}_+):=\mathfrak{H}_+,\quad \mathfrak{F}^{(1)}(\mathfrak{H}_-):=\Upsilon\mathfrak{H}_-.\tag{33}$$

To treat various numbers of particles and anti-particles, we recall the definition of the Fermi-Dirac Fock space:

$$\mathfrak{F}(\mathfrak{H}):=\bigoplus_{n,m=0}^{+\infty}\mathfrak{F}^{(n,m)},\quad \mathfrak{F}^{(n,m)}(\mathfrak{H}):=\mathfrak{F}^{(n)}(\mathfrak{H}_+)\otimes\mathfrak{F}^{(m)}(\mathfrak{H}_-),\tag{34}$$

where

$$\mathfrak{F}^{(0)}(\mathfrak{H}_+):=\mathbb{C},\quad \mathfrak{F}^{(0)}(\mathfrak{H}_-):=\mathbb{C},\quad \mathfrak{F}^{(n)}(\mathfrak{H}_+):=\bigwedge_{k=1}^{n}\mathfrak{H}_+,\quad \mathfrak{F}^{(m)}(\mathfrak{H}_-):=\bigwedge_{k=1}^{m}\Upsilon\mathfrak{H}_-.\tag{35}$$

An element ψ of $\mathfrak{F}(\mathfrak{H})$ consists of sequence $\psi=(\psi^{(n,m)})_{n,m\in\mathbb{N}}$, with $\psi^{(n,m)}\in\mathfrak{F}^{(n,m)}(\mathfrak{H})$. The vacuum vector is the vector $\Omega_{\text{vac}}\in\mathfrak{F}(\mathfrak{H})$ satisfying

$$(n,m)=(0,0)\Rightarrow\Omega^{(0,0)}_{\text{vac}}=1,\quad (n,m)\neq(0,0)\Rightarrow\Omega^{(n,m)}_{\text{vac}}=0.\tag{36}$$

We define the quantized Dirac field operator $\boldsymbol{\Psi}$ and its adjoint $\boldsymbol{\Psi}^*$:

$$f\in\mathfrak{H}\longmapsto\boldsymbol{\Psi}(f):=a(P_+f)+b^*(\Upsilon P_-f)\in\mathcal{L}(\mathfrak{H}),$$
$$f\in\mathfrak{H}\longmapsto\boldsymbol{\Psi}^*(f):=a^*(P_+f)+b(\Upsilon P_-f\in\mathcal{L}(\mathfrak{H}),$$

where $a(P_+f)$, $a^*(P_+f)$, $b(P_-f)$, $b^*(P_-f)$ are respectively the particle annihilation, creation operators and the anti-particle annihilation, creation operators. The quantized Dirac field is an anti-linear and bounded operator and, thanks to the classical properties of the creations and annihilations operators, it satisfies the canonical anti-commutation relations (CAR). We consider the C^*-algebra $\mathfrak{U}(\mathfrak{H})$ generated by the field operators $\boldsymbol{\Psi}^*(f)\boldsymbol{\Psi}(g)$, with $f,g\in\mathfrak{H}$. For an observable $A\in\mathfrak{U}(\mathfrak{H})$, we define the vacuum state as $\omega_{\text{vac}}(A):=<A\Omega_{\text{vac}},\Omega_{\text{vac}}>_{\mathfrak{H}}$. Then, by straightforward computation and for $f,g\in\mathfrak{H}$, we have

$$\omega_{\text{vac}}(\boldsymbol{\Psi}^*(f)\boldsymbol{\Psi}(g))=<P_-f,g>_{\mathfrak{H}}\ .\tag{37}$$

Given a Dirac-type equation, with Hamiltonian $\mathbb{H}$, satisfied by the one particle field f_D:

$$\partial_t f_D = i\mathbb{H} f_D\,,$$

we choose the spectral projectors P_+ and P_- such that

$$P_+ := \mathbf{1}_{]-\infty,0]}(\mathbb{H}), \quad P_- := \mathbf{1}_{[0,+\infty[}(\mathbb{H}). \tag{38}$$

On $\mathfrak{U}(\mathfrak{H})$, we also introduce the KMS state $\omega^{\delta,\sigma}_{\text{KMS}}$ depending on $\sigma > 0$ and $\delta \in \mathbb{R}$, such that for $f, g \in \mathfrak{H}$:

$$\omega^{\delta,\sigma}_{\text{KMS}}(\mathbf{\Psi}^*(f)\mathbf{\Psi}(g)) :=< \mu e^{\sigma\mathbb{H}}(1+\mu e^{\sigma\mathbb{H}})^{-1} f, g >_{\mathfrak{H}}, \quad \mu := e^{\sigma\delta}. \tag{39}$$

The restriction of this KMS state to the sub-algebra $\mathfrak{U}(\mathfrak{H}_+)$ (resp. $\mathfrak{U}(\mathfrak{H}_-)$) of $\mathfrak{U}(\mathfrak{H})$, corresponds to the Gibbs equilibrium state describing the thermodynamic models for noninteracting Fermi particles (resp. anti-particles) with temperature $\sigma^{-1} > 0$ and chemical potential δ (resp. $-\delta$).

As J. Dimock [9], we construct the algebra of local observables in the space-time outside the collapsing star, with the help of a given CAR representation on a Cauchy hyper-surface. In fact this construction does not depend on the choice of the CAR representation, the spin structure and the hyper-surface. Then, in particular, we consider the Fermi-Dirac Fock representation and the following foliation of the globally hyperbolic manifold:

$$\mathcal{M}_{\text{coll}} = \bigcup_{t\in\mathbb{R}} \Pi_t, \quad \Pi_t := \{t\}\times]z(t), +\infty[_{r_*}\times S^2_\omega.$$

We consider Π_0, and we put

$$\mathfrak{H} := L^2(]z(0), +\infty[\times S^2_\omega, r^2F^{1/2}(r)dr_*d\omega)^4 = \boldsymbol{L}^2_0, \quad \mathbb{H} := \boldsymbol{D}_0 \tag{40}$$

Using the previous definition of Dirac quantum field, we define on $\boldsymbol{L}^2_0$ the quantized Dirac field $\mathbf{\Psi}_0$ and $\mathfrak{U}(\mathfrak{H})$ the C^*-algebra generated by $\mathbf{\Psi}^*_0(\Phi_1)\mathbf{\Psi}_0(\Phi_2)$, with $\Phi_1, \Phi_2 \in \mathfrak{H}$. We introduce the following operator

$$S_{\text{coll}} : \Phi \in C^\infty_0(\mathcal{M}_{\text{coll}})^4 \longmapsto S_{\text{coll}}\Phi := \int_{\mathbb{R}} \boldsymbol{U}(0,t)\Phi(t)dt \in \boldsymbol{L}^2_0, \tag{41}$$

where $\boldsymbol{U}(0,t)$ is the propagator defined in proposition 3.1. Then, we define the local quantum field in $\mathcal{M}_{\text{coll}}$ by the operator:

$$\mathbf{\Psi}_{\text{coll}} : \Phi \in C^\infty_0(\mathcal{M}_{\text{coll}})^4 \longmapsto \mathbf{\Psi}_{\text{coll}}(\Phi) := \mathbf{\Psi}_0(S_{\text{coll}}\Phi), \tag{42}$$

and, for any open set $\mathcal{O} \subset \mathcal{M}_{\text{coll}}$, we introduce $\mathfrak{U}(\mathcal{O})$ the C^*-algebra generated by $\mathbf{\Psi}^*_{\text{coll}}(\Phi_1)\mathbf{\Psi}_{\text{coll}}(\Phi_2)$, $supp(\Phi_j) \subset \mathcal{O}$, $j = 1, 2$. Finally, we have:

$$\mathfrak{U}(\mathcal{M}_{\text{coll}}) = adh\left(\bigcup_{\mathcal{O}} \mathfrak{U}(\mathcal{O})\right).$$

Then, thanks to (37), (38) and (40), we define on $\mathfrak{U}(\mathcal{M}_{\text{coll}})$ a ground state $\omega_{\mathcal{M}_{\text{coll}}}$ as following:

$$\begin{aligned}\omega_{\mathcal{M}_{\text{coll}}}(\mathbf{\Psi}^*_{\text{coll}}(\Phi_1)\mathbf{\Psi}_{\text{coll}}(\Phi_2)) &:= \omega_{\text{vac}}(\mathbf{\Psi}^*_0(S_{\text{coll}}\Phi_1)\mathbf{\Psi}_0(S_{\text{coll}}\Phi_2)), \quad \Phi_1, \Phi_2 \in \mathfrak{H} \\ &=< \mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)S_{\text{coll}}\Phi_1, S_{\text{coll}}\Phi_2 >_{\mathfrak{H}}\end{aligned} \tag{43}$$

We describe the quantum field at the horizon of future back-hole. We consider the stationary space-time $\mathcal{M}_{\text{bh}}$ with the associated Dirac Hamiltonian $\boldsymbol{D}_{\leftarrow}$ for the one particle field. Using the Fermi-Dirac Fock quantization on $\mathbb{R}_{r_*} \times S^2_\omega$, we define the field $\mathbf{\Psi}_{\leftarrow}(\Phi)$ with $\Phi \in \boldsymbol{L}^2_{\leftarrow}$, and the operator $S_{\leftarrow}$ such that

$$S_{\leftarrow} : \Phi \in C^\infty_0(\mathcal{M}_{\text{bh}})^4 \longmapsto S_{\leftarrow}\Phi := \int_{\mathbb{R}} e^{-it\boldsymbol{D}_{\leftarrow}}\Phi(t)dt. \tag{44}$$

We also introduce

$$\boldsymbol{\Psi}_{\leftarrow} : \Phi \in C_0^\infty(\mathcal{M}_{\rm bh})^4 \longmapsto \boldsymbol{\Psi}_{\leftarrow}(\Phi) := \boldsymbol{\Psi}_{-}(S_{\leftarrow}\Phi), \tag{45}$$

and the C^*-algebra $\mathfrak{U}_{\leftarrow}(\mathcal{M}_{\rm bh})$ generated by $\boldsymbol{\Psi}_{\leftarrow}(\Psi_1)\boldsymbol{\Psi}^*_{\leftarrow}(\Psi_2)$, $\Phi_1, \Phi_2 \in \boldsymbol{L}^2_{\leftarrow}$. Using (39), we consider the Hawking thermal state:

$$\omega^{\delta,\sigma}_{\rm Haw}(\boldsymbol{\Psi}^*_{\leftarrow}(\Phi_1)\boldsymbol{\Psi}_{\leftarrow}(\Phi_2)) := \omega^{\delta,\sigma}_{\rm KMS}(\boldsymbol{\Psi}^*_{-}(S_{\leftarrow}\Phi_1)\boldsymbol{\Psi}_{-}(S_{\leftarrow}\Phi_2)), \quad \Phi_1,\Phi_2 \in C_0^\infty(\mathcal{M}_{\rm bh})^4 \tag{46}$$

$$=< \mu e^{\sigma \boldsymbol{D}_{\leftarrow}}(1+\mu e^{\sigma \boldsymbol{D}_{\leftarrow}})^{-1} S_{\leftarrow}\Phi_1, S_{\leftarrow}\Phi_2 >_{\boldsymbol{L}^2_{\leftarrow}}, \tag{47}$$

with

$$\mu := e^{\sigma\delta}, \quad \delta \in \mathbb{R}, \quad \sigma > 0. \tag{48}$$

Now, we describe the quantum field at the spatial infinity of the future black-hole. According to Λ which is respectively positive or zero (cosmological horizon or asymptotically flat space-time), we consider the stationary space-times $\mathcal{M}_{\rm bh}$ or $\mathcal{M}_{\rm flat} := \mathbb{R}_t \times \mathbb{R}^+_{r_*} \times S^2_\omega$, with the Dirac Hamiltonian associated to a one particle field $\boldsymbol{D}_{\Lambda,\rightarrow}$ and $\boldsymbol{D}_{0,\rightarrow}$. As above, using the Fermi-Dirac Fock quantization on $\mathbb{R}_{r_*} \times S^2_\omega$ or $\mathbb{R}^+_{r_*} \times S^2_\omega$, we define the fields $\boldsymbol{\Psi}_{\Lambda,+}(\Phi_1)$ with $\Phi_1 \in \boldsymbol{L}^2_{\Lambda,\rightarrow}$ or $\boldsymbol{\Psi}_{0,+}(\Phi_1)$ with $\Phi_1 \in \boldsymbol{L}^2_{0,\rightarrow}$ and the operators $S_{\Lambda,\rightarrow}$ or $S_{0,\rightarrow}$ characterized by:

$$S_{\Lambda,\rightarrow} : \Phi \in C_0^\infty(\mathcal{M}_{\rm bh})^4 \longmapsto S_{\Lambda,\rightarrow}\Phi := \int_{\mathbb{R}} e^{-it\boldsymbol{D}_{\Lambda,\rightarrow}}\Phi(t)dt, \quad \Lambda > 0, \tag{49}$$

$$S_{0,\rightarrow} : \Phi \in C_0^\infty(\mathcal{M}_{\rm flat})^4 \longmapsto S_{0,\rightarrow}\Phi := \int_{\mathbb{R}} \boldsymbol{U}_{0,\rightarrow}(-t)\Phi(t)dt, \tag{50}$$

where $\boldsymbol{U}_{0,\rightarrow}$ is the Dollard-modified propagator given by formula (30). Then, we construct the C^*-algebras $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\rm bh})$ and $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\rm flat})$, respectively generated by $\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\Phi_1)$ with $\Phi_1,\ \Phi_2 \in \boldsymbol{L}^2_{\Lambda,\rightarrow}$ and $\boldsymbol{\Psi}^*_{0,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{0,\rightarrow}(\Phi_1)$ with $\Phi_1,\Phi_2 \in \boldsymbol{L}^2_{0,\rightarrow}$, where

$$\boldsymbol{\Psi}_{\Lambda,\rightarrow} : \Phi \in C_0^\infty(\mathcal{M}_{\rm bh})^4 \longmapsto \boldsymbol{\Psi}_{\Lambda,\rightarrow}(\Phi) := \boldsymbol{\Psi}_{\Lambda,+}(S_{\Lambda,\rightarrow}\Phi), \quad \Lambda > 0, \tag{51}$$

$$\boldsymbol{\Psi}_{0,\rightarrow} : \Phi \in C_0^\infty(\mathcal{M}_{\rm flat})^4 \longmapsto \boldsymbol{\Psi}_{0,\rightarrow}(\Phi) := \boldsymbol{\Psi}_{0,+}(S_{0,\rightarrow}\Phi). \tag{52}$$

With (37), the vacuum states on each algebras $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\rm bh})$ and $\mathfrak{U}_{\rightarrow}(\mathcal{M}_{\rm flat})$ are given by

$$\omega_{\rm vac}(\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\Phi_1)) =< P^\Lambda_- S_{\Lambda,\rightarrow}\Phi_1, S_{\Lambda,\rightarrow}\Phi_2 >_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \quad \Lambda > 0, \tag{53}$$

$$\Phi_1,\Phi_2 \in C_0^\infty(\mathcal{M}_{\rm bh}), \quad P^\Lambda_- := \mathbf{1}_{[0,\infty[}(\boldsymbol{D}_{\Lambda,\rightarrow}), \tag{54}$$

$$\omega_{\rm vac}(\boldsymbol{\Psi}^*_{0,\rightarrow}(\Phi_1)\boldsymbol{\Psi}_{0,\rightarrow}(\Phi_1)) =< P^0_- S_{0,\rightarrow}\Phi_1, S_{0,\rightarrow}\Phi_2 >_{\boldsymbol{L}^2_{0,\rightarrow}}, \tag{55}$$

$$\Phi_1,\Phi_2 \in C_0^\infty(\mathcal{M}_{\rm flat}), \quad P^0_- := \mathbf{1}_{[0,\infty[}(\boldsymbol{D}_{0,\rightarrow}). \tag{56}$$

Since we are interested in the state of the quantum field at the last moment of gravitational collapse, we investigate the following limit:

$$\lim_{T\to+\infty} \omega_{\mathcal{M}_{\rm coll}}(\boldsymbol{\Psi}^*_{\rm coll}(\Phi_1^T)\boldsymbol{\Psi}_{\rm coll}(\Phi_2^T)),$$

where

$$\Phi_j^T(t,r_*,\omega) := \Phi_j(t-T,r_*,\omega), \quad \Phi_j \in C_0^\infty(\mathcal{M}_{\rm coll})^4, \quad j = 1,2,$$

and, $\omega_{\mathcal{M}_{\rm coll}}$ and $\boldsymbol{\Psi}_{\rm coll}$ are defined by (43) and (42). Then, we state the main theorem of this work

Theorem 5.1 *(Main result)*
Given $\Phi_j \in C_0^\infty(\mathcal{M}_{coll})^4$, $\quad j = 1, 2$, *then we have for* $\Lambda \geq 0$,

$$\lim_{T\to+\infty} \omega_{\mathcal{M}_{coll}}(\boldsymbol{\Psi}^*_{coll}(\Phi_1^T)\boldsymbol{\Psi}_{coll}(\Phi_2^T)) = \omega^{\delta,\sigma}_{Haw}(\boldsymbol{\Psi}^*_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi_1)\boldsymbol{\Psi}_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi_2)) + \omega_{vac}(\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi_1)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi_2)),$$

with

$$T_{Haw} = \frac{1}{\sigma} = \frac{2\pi}{\kappa_0}, \quad \delta = \frac{qQ}{r_0}.$$

Proof of theorem 5.1 :
For $\Phi \in C_0^\infty(\mathcal{M}_{\text{coll}})^4$, by the identity of polarization, it is sufficient to evaluate

$$\begin{aligned}\lim_{T\to+\infty} \omega_{\mathcal{M}_{\text{coll}}}(\boldsymbol{\Psi}^*_{\text{coll}}(\Phi^T)\boldsymbol{\Psi}_{\text{coll}}(\Phi^T)) &= \lim_{T\to+\infty}\left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\ S_{\text{coll}}\Phi^T\right\|_0^2, \\ &= \lim_{T\to+\infty}\left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\ \boldsymbol{U}(0,T)S_{\text{bh}}\Phi\right\|_0^2, \end{aligned} \tag{57}$$

because for $T > 0$ large enough, we have:

$$S_{\text{coll}}\Phi^T = \boldsymbol{U}(0,T)S_{\text{bh}}\Phi, \quad S_{\text{bh}}\Phi := \int_{\mathbb{R}} \boldsymbol{U}(-t)\Phi(t)dt.$$

Then, we use the key theorem that we prove in the next section:

Theorem 5.2
Given $f \in \boldsymbol{L}^2_{BH}$, *if* $\Lambda \geq 0$, *then*

$$\begin{aligned}\lim_{T\to+\infty}\left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\ \boldsymbol{U}(0,T)f\right\|_0^2 &= \left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\Lambda,\rightarrow})\boldsymbol{\Omega}^-_{\Lambda,\rightarrow} f\right\|^2_{\boldsymbol{L}^2_{\Lambda,\rightarrow}} \\ &+ < \boldsymbol{\Omega}^-_{\leftarrow}f, \mu e^{\sigma\boldsymbol{D}_{\leftarrow}}\left(1+\mu e^{\sigma\boldsymbol{D}_{\leftarrow}}\right)^{-1}\boldsymbol{\Omega}^-_{\leftarrow}f >_{\boldsymbol{L}^2_{\leftarrow}}\end{aligned} \tag{58}$$

with

$$\mu = e^{\sigma\delta}, \quad \delta := \frac{qQ}{r_0} \quad \sigma = \frac{2\pi}{\kappa_0}, \quad \boldsymbol{\Omega}^-_{\leftarrow} := \left(\boldsymbol{W}^-_{\leftarrow}\right)^*, \quad \boldsymbol{\Omega}^-_{\Lambda,\rightarrow} := \left(\boldsymbol{W}^-_{\Lambda,\rightarrow}\right)^*, \quad \boldsymbol{\Omega}^-_{0,\rightarrow} := \left(\boldsymbol{W}^-_{0,\rightarrow}\right)^*,$$

where $\boldsymbol{W}^-_{\leftarrow}$, $\boldsymbol{W}^-_{\Lambda,\rightarrow}$, $\boldsymbol{W}^-_{0,\rightarrow}$ *are the wave operators respectively defined in (28), (29) and (31).*

According to (57) and the previous theorem, for $\Lambda \geq 0$, we deduce that :

$$\begin{aligned}\lim_{T\to+\infty} \omega_{\mathcal{M}_{\text{coll}}}(\boldsymbol{\Psi}^*_{\text{coll}}(\Phi^T)\boldsymbol{\Psi}_{\text{coll}}(\Phi^T)) &= \left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\Lambda,\rightarrow})\boldsymbol{\Omega}^-_{\Lambda,\rightarrow} S_{\text{bh}}\Phi\right\|^2_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \\ &+ < \boldsymbol{\Omega}^-_{\leftarrow}S_{\text{bh}}\Phi, \mu e^{\sigma\boldsymbol{D}_{\leftarrow}}\left(1+\mu e^{\sigma\boldsymbol{D}_{\leftarrow}}\right)^{-1}\boldsymbol{\Omega}^-_{\leftarrow}S_{\text{bh}}\Phi >_{\boldsymbol{L}^2_{\leftarrow}} \\ &= \left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\Lambda,\rightarrow})S_{\Lambda,\rightarrow}\ \boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\ \Phi\right\|^2_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \\ &+ < S_{\leftarrow}\ \boldsymbol{\Omega}^-_{\leftarrow}\Phi, \mu e^{\sigma\boldsymbol{D}_{\leftarrow}}\left(1+\mu e^{\sigma\boldsymbol{D}_{\leftarrow}}\right)^{-1}S_{\leftarrow}\ \boldsymbol{\Omega}^-_{\leftarrow}\Phi >_{\boldsymbol{L}^2_{\leftarrow}} \\ &= \omega^{\delta,\sigma}_{\text{Haw}}(\boldsymbol{\Psi}^*_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi_1)\boldsymbol{\Psi}_{\leftarrow}(\boldsymbol{\Omega}^-_{\leftarrow}\Phi_2)) \\ &\quad + \omega_{\text{vac}}(\boldsymbol{\Psi}^*_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi)\boldsymbol{\Psi}_{\Lambda,\rightarrow}(\boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\Phi)).\end{aligned}$$

■

5.2 Discussion

The interpretation of the previous theorem in terms of particles is more difficult. Indeed, there are as many definitions of particles as types of observators. In the Minkowski space time and thanks to the Lorentz transformations, we naturally define the particles linked to the inertial observators. For the general curved space-times, we have not the similar transformations and the notion of particles is rather vague. In Theorem 5.1, the state $\omega_{\mathcal{M}_{\text{coll}}}(\mathbf{\Psi}^*_{\text{coll}}(\Phi^T)\mathbf{\Psi}_{\text{coll}}(\Phi^T))$ gives informations at the time T of a detector fixed with the respect to the variables (r_*, ω) measuring the fluctuation of the quantum field outside the collapsing star. The detector is put in the Boulware vacuum that corresponds to the classical concept of vacuum state for a static observer. This last theorem gives the response of the detector at their own infinite proper time ($T = +\infty$), which corresponds to the last moments of gravitational collapse. On the hand, the term $\omega_{\text{vac}}(\mathbf{\Psi}^*_{\Lambda,\rightarrow}(\mathbf{\Omega}^-_{\Lambda,\rightarrow}\Phi_1)\mathbf{\Psi}_{\Lambda,\rightarrow}(\mathbf{\Omega}^-_{\Lambda,\rightarrow}\Phi_2))$ proves that the dectector measures merely a vacuum coming from the past infinity and falling into the black hole. On the other hand, $\omega^{\delta,\sigma}_{\text{Haw}}(\mathbf{\Psi}^*_{\leftarrow}(\mathbf{\Omega}^-_{\leftarrow}\Phi_1)\mathbf{\Psi}_{\leftarrow}(\mathbf{\Omega}^-_{\leftarrow}\Phi_2))$ corresponds to the emergence of a thermal state at temperature T_{Haw} coming from the vicinity of the black hole. An observer at rest with respect to coordinates (r_*, ω) will interpret as $t \to +\infty$ this thermal state like a flux of fermionic and anti-fermionics particles leaving the future black hole. The result is independent of the history of the collapse and the boundary condition on the star surface. Indeed, we can easily prove the same theorem putting the more general *MIT Bag* boundary condition (see [4]):

$$\mathcal{B} := i \sum_{(ln)\in\mathcal{I}} e^{i\nu_{l,n}\gamma^5} P_{ln}$$

where P_{ln} is the orthogonal $L^2(S^2_\omega)$-projector on $Vect(Y_{ln})$, (see (68)) and $\nu_{l,n}$ a sequence which satisfies the same conditions as ν in the third section about the *MIT Bag* boundary condition. Moreover, for a Lebesgue measurable subset B of $\mathbb{R}_{r_*} \times S^2_\omega$ with $0 < |B| < +\infty$, lemma A.2 in [4] gives respectively the expression of the density of particles $D^+_B(\omega^{\delta,\sigma}_{\text{KMS}})$, of antiparticles $D^-_B(\omega^{\delta,\sigma}_{\text{KMS}})$ and the charge density ρ_{Haw} for the gas of fermions create at the vicinity of the black-hole horizon in the subset B:

$$D^+_B(\omega^{\delta,\sigma}_{\text{KMS}}) := B^{-1} \sum \omega^{\delta,\sigma}_{\text{KMS}}(a^*(P^+_{\leftarrow}\Phi^j)a(P^+_{\leftarrow}\Phi^j)) = \frac{1}{\pi\sigma}\ln(1+e^{\sigma\delta}), \tag{59}$$

$$D^-_B(\omega^{\delta,\sigma}_{\text{KMS}}) := B^{-1} \sum \omega^{\delta,\sigma}_{\text{KMS}}(a^*(P^-_{\leftarrow}\Phi^j)a(P^-_{\leftarrow}\Phi^j)) = \frac{1}{\pi\sigma}\ln(1+e^{-\sigma\delta}), \tag{60}$$

$$P^+_{\leftarrow} := \mathbf{1}_{]-\infty,0]}(\boldsymbol{D}_{\leftarrow}), \quad P^-_{\leftarrow} := \mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\leftarrow}), \tag{61}$$

$$\rho_{\text{Haw}} := q\left(D^+_B(\omega^{\delta,\sigma}_{\text{KMS}}) + D^-_B(\omega^{\delta,\sigma}_{\text{KMS}})\right) = \frac{1}{\pi}q\delta = \frac{q^2Q}{\pi r_0}, \tag{62}$$

where $(\Phi^j)_{j\in\mathbb{N}}$ is an orthonormal basis of $\{S_{\leftarrow}\mathbf{\Omega}^-_{\leftarrow}\Phi \in \boldsymbol{L}^2_{\text{BH}} : (r_*,\omega) \notin B \Rightarrow S_{\leftarrow}\mathbf{\Omega}^-_{\leftarrow}\Phi(r_*,\omega) = 0\}$. Since ρ_{Haw} and Q have the same sign, we conclude that the black-hole preferentially emits charged particles with the same sign as its own charge.

We emphasize that the interpretation of theorem 5.1 is valid only in semiclassical regime. Indeed, we suppose that the black hole that we consider has a sufficiently large mass in order to be able to use the classical theory of General Relativity to model the gravitational field but also to neglect the back reaction of the quantum fields. Thanks to theorem 5.1, we can conjecture that the black hole loses its charge and its mass. Therefore, if we want to study this evaporation, we can not neglect the back reaction of the Hawking effect. But for that, it would be necessary to study a non linear problem of a very great complexity.

6 Proofs of the main theorems.

This section is organized as follow: in the first subpart, thanks to the spherical symmetry property of the geometrical framework, we reduce (17) and (20) to a family of one dimensional problems. This reduction will be useful for the next subparts. In the second part, we prove theorem 4.1 on the scattering theory in the eternal charged black-hole. In the third part, we demonstrate theorem 5.2 on the sharp estimate of $\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)$.

6.1 Reduction to a one dimensional problem.

To reduce problems (17) and (20), we use spin-weighted harmonics $Y^l_{\pm\frac{1}{2},n}$ (see [12], [17]). The families

$$\left\{Y^l_{\frac{1}{2},n};\ (l,n)\in\mathcal{I}\right\},\quad \left\{Y^l_{-\frac{1}{2},n};\ (l,n)\in\mathcal{I}\right\},\quad \mathcal{I}:=\left\{(l,n):\ \ l-\frac{1}{2}\in\mathbb{N},\ \ l-|n|\in\mathbb{N}\right\},$$

form a Hilbert basis of $L^2(S^2_\omega)$ and each Y^l_{sn}, $s=\pm 1/2$ satisfies the recurrence relations,

$$\partial_\theta Y^l_{sn}(\omega)\mp\frac{n-s\cos\theta}{\sin\theta}Y^l_{sn}(\omega)=\left|\begin{array}{l} -i\sqrt{(l\pm s)(l\mp s+1)}Y^l_{s\mp 1,n}(\omega),\quad \pm l>-s. \\ 0,\quad l=\mp s. \end{array}\right. ,\tag{63}$$

$$\partial_\varphi Y^l_{sn}(\omega)=-inY^l_{sn}(\omega).\tag{64}$$

We introduce the Hilbert spaces to treat the one dimensional problem respectively outside, the charged collapsing star and the eternal black hole:

$$0\le t,\quad L^2_t:=L^2(]z(t),+\infty[_{r_*},\ dr_*)^4,\quad L^2_{\mathbb{R}}:=L^2(\mathbb{R}_{r_*},\ dr_*)^4,\quad L^2_{\text{BH}}:=L^2(\mathbb{R}_{r_*},\ r^2F^{1/2}(r)dr_*)^4.\tag{65}$$

The norm of L^2_t and $L^2_{\mathbb{R}}$ are respectively denoted by $\|.\|_t$ and $\|.\|$. Moreover for $\Phi\in L^2(B,\ dr_*)^4$, $B\subset\mathbb{R}$,

$$\|\Phi\|_{L^2(B,\ dr_*)^4}=\|[\Phi]_L\|,\quad [\Phi]_L(r_*):=\left\{\begin{array}{ll}\Phi(r_*) & r_*\in B\\ 0 & r_*\in\mathbb{R}\setminus B\end{array}\right. .$$

In the same way, we define

$$0\le t,\quad H^1_t:=\left\{\Phi\in L^2_t,\ \partial_{r_*}\Phi\in L^2_t\right\},\quad H^1_{\mathbb{R}}:=\left\{\Phi\in L^2_{\mathbb{R}},\ \partial_{r_*}\Phi\in L^2_{\mathbb{R}}\right\},$$

and moreover for $\Phi\in H^1_t$ we have,

$$[\Phi]_H\in H^1_{\mathbb{R}},\quad [\Phi]_H(r_*):=\left\{\begin{array}{ll}\Phi(r_*) & r_*\in]z(t),+\infty[_{r_*}\\ \Phi(2z(t)-r_*) & r_*\in\mathbb{R}\setminus]z(t),+\infty[_{r_*}\end{array}\right. .$$

Hence, for $0\le t\le+\infty$, and putting

$$\mathcal{P}_r:\Psi\mapsto r^{-1}F^{-1/4}\Psi,\tag{66}$$

any $\Psi\in\boldsymbol{L}^2_t$ or $\boldsymbol{L}^2_{\text{BH}}$, where $\Psi_{ln}\in\mathcal{P}_rL^2_t$ or $\mathcal{P}_rL^2_{\mathbb{R}}$ can be written in the following way:

$$\Psi(r_*,\omega)=\sum_{(l,n)\in\mathcal{I}}\Psi_{ln}(r_*)\otimes_4 Y_{ln}(\omega),\tag{67}$$

$$v\otimes_4 u:=(u_1v_1,u_2v_2,u_3v_3,u_4v_4),\quad \forall u,v\in\mathbb{C}^4,$$

$$Y_{ln}:=\left(Y^l_{-\frac{1}{2},n},Y^l_{\frac{1}{2},n},Y^l_{-\frac{1}{2},n},Y^l_{\frac{1}{2},n}\right).\tag{68}$$

We define,

$$\mathcal{R}^\nu_{ln}:\Psi\in\boldsymbol{L}^2_t\mapsto e^{i\frac{\nu}{2}\gamma^5}\mathcal{P}_r^{-1}\Psi_{ln}\in L^2_t,\tag{69}$$

$$\mathcal{R}^{\text{BH}}_{ln}:\Psi\in\boldsymbol{L}^2_{\text{BH}}\mapsto\Psi_{ln}\in\mathcal{P}_rL^2_{\mathbb{R}}\tag{70}$$

$$\mathcal{E}^\nu_{ln}:\Psi_{ln}\in L^2_t\mapsto e^{-i\frac{\nu}{2}\gamma^5}\mathcal{P}_r\Psi_{ln}\otimes_4 Y_{ln}\in\boldsymbol{L}^2_t,\tag{71}$$

$$\mathcal{E}^{\text{BH}}_{ln}:\Psi_{ln}\in\mathcal{P}_rL^2_{\mathbb{R}}\mapsto\Psi_{ln}\otimes_4 Y_{ln}\in\boldsymbol{L}^2_{\text{BH}}.\tag{72}$$

to express $\boldsymbol{L}_t^2$ and $\boldsymbol{L}_{\text{BH}}^2$ as a direct sum:

$$\boldsymbol{L}_t^2 = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^\nu L_t^2, \quad \boldsymbol{L}_{\text{BH}}^2 = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^{\text{BH}} L_{\text{BH}}^2 = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^\nu L_{\mathbb{R}}^2. \tag{73}$$

With (63), (64) and $s = \pm 1/2$, we obtain

$$\boldsymbol{D}_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^\nu D_{V_{l,\nu},t} \mathcal{R}_{ln}^\nu - \frac{qQ}{r_0}, \tag{74}$$

$$D_{V_{l,\nu},t} := \Gamma^1 \partial_{r_*} + V_{l,\nu}, \quad V_{l,\nu} = qQ\left(\frac{1}{r_0} - \frac{1}{r}\right) - \sqrt{F(r)}\left(mA_\nu + \frac{i}{r}\Gamma^2(l+1/2)\right), \tag{75}$$

$$A_\nu := \begin{pmatrix} 0 & a_\nu \\ \bar{a_\nu} & 0 \end{pmatrix}, \quad a_\nu := \text{diag}(ie^{i\nu}, ie^{i\nu}), \quad Z(t) = \sqrt{\frac{1-\dot{z}(t)}{1+\dot{z}(t)}}, \tag{76}$$

$$\mathcal{D}(D_{V_{l,\nu},t}) = \{\Psi \in L_t^2;\ D_{V_{l,\nu},t}\Psi \in L_t^2,$$
$$Z(t)\Psi_2(z(t)) = \Psi_4(z(t)),\ \Psi_1(z(t)) = -Z(t)\Psi_3(z(t))\} \tag{77}$$

and

$$\boldsymbol{D}_{\text{BH}} = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}_{ln}^{\text{BH}} D_{\text{BH}} \mathcal{R}_{ln}^{\text{BH}}, \quad D_{\text{BH}} = \Gamma^1\left(\partial_{r_*} + \frac{F(r)}{r} + \frac{1}{4}F(r)\right) + V_{\text{BH}} \tag{78}$$

$$V_{\text{BH}} = -\frac{qQ}{r} - \sqrt{F(r)}\left(\frac{i}{r}\Gamma^2(l+1/2) - \Gamma^4\right), \tag{79}$$

$$\mathcal{D}(D_{\text{BH}}) = \{\Psi \in L_{\text{BH}}^2;\ D_{\text{BH}}\Psi \in L_{\text{BH}}^2\}. \tag{80}$$

Therefore, Ψ is solution of problem (17), (18) and (19) if and only if, for all $(l,n) \in \mathcal{I}$,

$$\Phi(t, r_*) := e^{itqQr_0^{-1}} \mathcal{R}_{ln}^\nu \Psi(t, r_*)$$

is solution of

$$\partial_t \Phi = iD_{V_{l,\nu},t}\Phi, \quad t \in \mathbb{R}, \quad r_* > z(t), \tag{81}$$

$$Z(t)\Phi_2(t, z(t)) = \Phi_4(t, z(t)), \quad -Z(t)\Phi_3(t, z(t)) = \Phi_1(t, z(t)), \tag{82}$$

$$\Phi(t = s, .) = \Phi_s(.) := \mathcal{R}_{ln}^\nu \Psi_s(.) \in L_s^2. \tag{83}$$

In the same way, Ψ is solution of problem (20) and (21) if and only if, for all $(l,n) \in \mathcal{I}$,

$$\Phi(t, r_*) := \mathcal{R}_{ln}^{\text{BH}} \Psi(t, r_*)$$

is solution of

$$\partial_t \Phi = iD_{\text{BH}}\Phi, \tag{84}$$

$$\Phi(t = 0, .) = \Phi_{\text{BH}} := \mathcal{R}_{ln}^{\text{BH}} \Psi_{\text{BH}} \in L_{\text{BH}}^2. \tag{85}$$

In [4], proposition VI.2 gives a solution $\Phi(t)$ of the problem (81), (82) and (83) expressed with the propagator $U_{V_{l,\nu}}(t, s)$:

Proposition 6.1
If $\Phi_s \in \mathcal{D}(D_{V_{l,\nu},s})$, then there exists a unique solution $[\Phi(.)]_H = [U_{V_{l,\nu}}(., s)\Phi_s]_H \in C^1(\mathbb{R}_t, L_{\mathbb{R}}^2) \cap C^0(\mathbb{R}_t, H_{\mathbb{R}}^1)$ of (81), (82) and (83) :

$$\Phi(t) \in \mathcal{D}(D_{V_{l,\nu},t}).$$

Moreover,

$$\|\Phi(t)\|_t = \|\Phi_s\|_s \tag{86}$$

and $U_{V_{l,\nu}}(t,s)$ *can be extended in an isometric strongly continuous propagator from* L^2_s *onto* L^2_t*, and for an* $R > z(s)$

$$(x > R \Rightarrow \Phi_s(r_*,\omega) = 0) \Rightarrow (x > R + |t-s| \Rightarrow [U_{V_{l,\nu}}(t,s)\Phi_s](r_*,\omega) = 0).$$

Thanks to the notations (69) and (71), we give the important relations connecting propagator $U_V(t,s)$ with $\boldsymbol{U}(t,s)$ defined in proposition (3.1):

$$\boldsymbol{U}(t,s) = e^{i(s-t)\frac{qQ}{r_0}} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln} U_{V_{l,\nu}}(t,s)\mathcal{R}^{\nu}_{ln} : \boldsymbol{L}^2_s = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln} L^2_s \to \boldsymbol{L}^2_t = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln} L^2_t. \tag{87}$$

Subsequently, to simplify the notations, we forget subscripts ln and ν in the above one dimensional problem. Given a interval $B := (a,b) \subset \mathbb{R}_{r_*}$ and $V \in L^{\infty}(\mathbb{R}_{r_*})$, then, on $L^2(B)^4$ we define the self-adjoint operator $D_{V,B}$ with the dense domain $\mathcal{D}(D_{V,B})$ such that

$$D_{V,B} = \Gamma^1 \partial_{r_*} + V, \tag{88}$$

$$\mathcal{D}(D_{V,B}) = \left\{\Phi \in L^2(B)^4;\ D_{V,B}\Phi \in L^2(B)^4,\ r_* \in \partial B \Rightarrow \vec{n}\gamma^1\Phi(r_*) = i\Phi(r_*)\right\}, \tag{89}$$

where $\vec{n}$ is the outgoing normal of B and Γ^1 given by (23). Hence by the Kato-Rellich and spectral theorem, the problem

$$\partial_t \Phi = iD_{V,B}\Phi, \quad \Phi(t=0)\Psi_0, \tag{90}$$

is solved with the help of the propagator $U_{V,B}(t)$, following the proposition:

Proposition 6.2
Given $\Phi_0 \in \mathcal{D}(D_{V,B})$*, then there exists a unique solution* $\Phi(.) = U_{V,B}(.)\Phi_0 \in C^0(\mathbb{R}_t, \mathcal{D}(D_{V,B})) \cap C^1(\mathbb{R}_t, L^2(B)^4)$ *and*

$$\|\Phi(t)\| = \|\Phi_0\|.$$

Moreover, $U_{V,B}(t)$ *can be extended, by density and continuity, in strongly unitary group on* $L^2(B)^4$.

In some useful particular cases, we have an explicit formula:

Lemma 6.1
Given $\Phi^0 = (\Phi^0_1, \Phi^0_2, \Phi^0_3, \Phi^0_4) \in L^2_s$ *for* $t \geq s$*, then* $\Phi(t,r_*) = U_0(t,s)\Phi_0(r_*)$ *is given by*

$$r_* > z(t):\ \Phi_2(t,r_*) = \Phi^0_2(r_* - t + s), \quad \Phi_3(t,r_*) = \Phi^0_3(r_* - t + s),$$
$$r_* > z(t) + s - t:\ \Phi_1(t,r_*) = \Phi^0_1(r_* + t - s), \quad \Phi_4(t,r_*) = \Phi^0_3(r_* + t - s),$$
$$z(t) < r_* < z(t) + s - t:\ \Phi_1(t,r_*) = -Z(\tau(r_* + t))\Phi^0_3(r_* + t + s - 2\tau(r_* + t)),$$
$$z(t) < r_* < z(t) + s - t:\ \Phi_4(t,r_*) = Z(\tau(r_* + t))\Phi^0_2(r_* + t + s - 2\tau(r_* + t)),$$

where τ *is defined by (7). Given* $\Phi^0 \in L^2(B)^4$*, with* $B =]-\infty, a]$ *or* $[a, +\infty[$*,* $a \in \mathbb{R} \cup \{-\infty, +\infty\}$ *and* $\delta \in \mathbb{R}$*, then, if* $B =]-\infty, a]$*,* $\Phi(t,r_*) = U_{\delta,B}(t)\Phi_0(r_*)$ *is given by*

$$\Phi(t,r_*) = \begin{cases} e^{i\delta t} \ {}^t\left(\Phi^0_3(2a - r_* - t), \Phi^0_2(r_* - t), \Phi^0_3(r_* - t), -\Phi^0_2(2a - r_* - t)\right), & r_* + t \geq a, \\ e^{i\delta t} \ {}^t\left(\Phi^0_1(r_* + t), \Phi^0_2(r_* - t), \Phi^0_3(r_* - t), \Phi^0_4(r_* + t)\right), & r_* + t \leq a,\ r_* - t \leq a, \\ e^{i\delta t} \ {}^t\left(\Phi^0_1(r_* + t), -\Phi^0_4(2a - r_* + t), \Phi^0_1(2a - r_* + t), \Phi^0_4(r_* + t)\right), & r_* - t \geq a, \end{cases}$$

and, if $B = [a, +\infty[$*, by*

$$\Phi(t,r_*) = \begin{cases} e^{i\delta t} \ {}^t\left(\Phi^0_1(r_* + t), \Phi^0_4(2a + t - r_*), -\Phi^0_1(2a + t - r_*), \Phi^0_4(r_* + t)\right), & r_* - t \leq a, \\ e^{i\delta t} \ {}^t\left(\Phi^0_1(r_* + t), \Phi^0_2(r_* - t), \Phi^0_3(r_* - t), \Phi^0_4(r_* + t)\right), & r_* - t \geq a,\ r_* + t \geq a, \\ e^{i\delta t} \ {}^t\left(-\Phi^0_3(2a - r_* - t), \Phi^0_2(r_* - t), \Phi^0_3(r_* - t), \Phi^0_2(2a - r_* - t)\right), & r_* + t \leq a. \end{cases}$$

Proof:
The result follows from the study of the characteristics of problems (81)-(82) and (84). ∎

6.2 Proof of theorem 4.1 on the scattering theory

Before proving theorem 4.1, we state the following proposition concerning the spectral properties of D_{BH} and $\boldsymbol{D}_{\mathrm{BH}}$.

Proposition 6.3
If $\Lambda \geq 0$, then

$$\sigma(D_{BH}) = \sigma_{ac}(D_{BH}) = \mathbb{R} \tag{91}$$

and

$$\sigma(\boldsymbol{D}_{BH}) = \sigma_{ac}(\boldsymbol{D}_{BH}) = \mathbb{R}. \tag{92}$$

with D_{BH} and $\boldsymbol{D}_{BH}$ given by (78)(80) and (22)(25).

Proof:
When $\Lambda = 0$ the properties (91) and (92) have been proved in [19]. If $\Lambda > 0$ the proof remains essentially similar. Principally, our demonstration in [19] bases one's argument on the Mourre theory and, in this work, when $\Lambda = 0$, we wrote

$$\begin{aligned} -D_{\mathrm{BH}} &= -\Gamma^1 \partial_{r_*} + V_q + V_l + V_m \\ V_q := \frac{qQ}{r}, \quad V_l &:= -(l+1/2)\Gamma^2 \sqrt{F(r)}\frac{i}{r}, \quad V_m := \sqrt{F(r)}\Gamma^4 = m\sqrt{F(r)}\gamma^0. \end{aligned}$$

The main difficulty of this proof is the obtaining of Mourre inequality. To do this, we must choose an appropriate conjugate operator A. But, we remark that

$$\lim_{r_* \to -\infty} V_q = \frac{qQ}{r_0}.$$

For the positive energies, when $qQ < 0$ (respectively for negative energies and $qQ > 0$), we obtain easily this inequality if A is the classical generator of dilations. But, when $qQ > 0$ (respectively $qQ < 0$), this choice of conjugate operator does not allows us to obtain the result. Indeed, if we put $h = -D_{\mathrm{BH}}$ and consider the case $qQ > 0$, then we obtain the following equality (in sense of the quadratic forms in $H^1_{\mathbb{R}}$):

$$\chi(h) i[h, A]\chi(h) \geq (\varepsilon - qQr^{-1})\chi^2(h) + k, \quad \varepsilon > 0, \quad A := -\frac{i}{2}\left(r_*\partial_{r_*} + \partial_{r_*} r_*\right),$$

where k is a $L^2_{\mathbb{R}}$ compact operator and $\chi \in C_0^\infty(\mathbb{R})$ such that $supp\chi \subset \mathbb{R}_*^+ - \{m\}$. Then, to overcome the problem, we put:

$$A := -\frac{i}{2}\left(r_*\partial_{r_*} + \partial_{r_*} r_*\right) + \frac{qQ}{r_0}\gamma^0\gamma^1 r_* J_-(r_*), \quad J_- \in C^\infty(\mathbb{R}_{r_*}), \quad J_-(r_*) = \left\{ \begin{array}{ll} 1 & r_* \leq -3 \\ 0 & r_* \geq -2 \end{array} \right. .$$

With this choice, the Mourre assumptions are satisfied and since $qQr_0^{-1} - qQr^{-1} \geq 0$, we have:

$$\chi(h) i[h, A]\chi(h) \geq (\varepsilon + qQr_0^{-1}J_- - qQr^{-1}J_-)\chi^2(h) + k^{'} \geq \varepsilon\chi^2(h) + k^{'},$$

with $\varepsilon > 0$ and $k^{'}$ is a compact operator on $L^2_{\mathbb{R}}$. When $\Lambda > 0$, the result becomes widespread. Indeed, we put

$$\widetilde{h} := h - \frac{qQ}{r_+}.$$

Then, for the difficult cases, we define

$$A := -\frac{i}{2}\left(r_*\partial_{r_*} + \partial_{r_*} r_*\right) + qQ\left(\frac{1}{r_0} - \frac{1}{r_+}\right)\gamma^0\gamma^1 r_* J_-(r_*).$$

Therefore, for $qQ>0$ and $supp\chi \subset \mathbb{R}_*^+ - \{m\}$, we obtain:

$$\begin{aligned}\chi(h)i[h,A]\chi(h) &\geq \varepsilon\chi^2(h) + qQJ_-(r_0^{-1} - qQr_+^{-1})\chi^2(h) - qQJ_-(r^{-1} - qQr_+^{-1})\chi^2(h) + k'' \\ &\geq \varepsilon\chi^2(h) + k'', \quad \varepsilon > 0,\end{aligned}$$

with $\varepsilon > 0$ and k'' is a $L^2_{\mathbb{R}}$-compact operator on $L^2_{\mathbb{R}}$. To finish, as in [4], we check that D_{BH} has no eigenvalues when $\Lambda \geq 0$. ■

Proof of theorem 4.1:
The case where $\Lambda = 0$ was proved in [19] and we consider only the case $\Lambda > 0$. Given two self-adjoint operators A on $\mathcal{H}_A$ and B on $\mathcal{H}_B$, we formally define the wave operators

$$W^{\pm}(A,B,\mathcal{J}) = s - \lim_{t\to\pm\infty} e^{-itA}\mathcal{J}e^{itB}P_{ac}(B), \tag{93}$$

where $P_{ac}(B)$ is the projector on the absolutely continuous subspace of B and $\mathcal{J}$ the bounded identifying operator between $\mathcal{H}_B$ and $\mathcal{H}_A$. When $\mathcal{H}_A = \mathcal{H}_B$ and $\mathcal{J} = Id$, we denote $W^{\pm}(A,B,Id)$ simply by $W^{\pm}(A,B)$. First, we separate the problems at the horizon and at infinity. To do this, we use the self-adjoint operator $\boldsymbol{D}^-_{\text{BH}} \oplus \boldsymbol{D}^+_{\text{BH}}$ on $\boldsymbol{L}^2_{\text{BH}}$, such that :

$$\begin{aligned}
&\boldsymbol{D}^-_{\text{BH}}, \boldsymbol{D}^+_{\text{BH}} := -\frac{qQ}{r} + \Gamma^1\left(\partial_{r_*} + \frac{F(r)}{r} + \frac{1}{4}F(r)\right) + \sqrt{F(r)}\left(\frac{\Gamma^2}{r}(\partial_\theta + \frac{1}{2}\cot\theta) + \frac{\Gamma^3}{r\sin\theta}\partial_\varphi + \Gamma^4\right),\\
&\mathcal{D}\left(\boldsymbol{D}^-_{\text{BH}}\right) = \Big\{\Psi \in L^2(]-\infty,1]_{r_*} \times S^2_\omega, r^2F^{1/2}(r)dr_*d\omega)^4;\\
&\qquad\qquad \boldsymbol{D}^-_{\text{BH}}\Psi \in L^2(]-\infty,1]_{r_*} \times S^2_\omega, r^2F^{1/2}(r)dr_*d\omega)^4, \gamma^1\Psi(1,.) = i\Psi(1,.)\Big\},\\
&\mathcal{D}\left(\boldsymbol{D}^+_{\text{BH}}\right) = \Big\{\Psi \in L^2([1,+\infty[_{r_*}\times S^2_\omega, r^2F^{1/2}(r)dr_*d\omega)^4;\\
&\qquad\qquad \boldsymbol{D}^+_{\text{BH}}\Psi \in L^2([1,+\infty[_{r_*}\times S^2_\omega, r^2F^{1/2}(r)dr_*d\omega)^4, -\gamma^1\Psi(1,.) = i\Psi(1,.)\Big\}.
\end{aligned}$$

Thanks to formula (78) we reduce $\boldsymbol{D}_{\text{BH}}$ on $\boldsymbol{L}^2_{\text{BH}}$ by D_{BH} on L^2_{BH}. In the same way, via operators (70) and (72), we can also reduce $\boldsymbol{D}^-_{\text{BH}} \oplus \boldsymbol{D}^+_{\text{BH}}$ on $\boldsymbol{L}^2_{\text{BH}}$ by the self-adjoint operator $D^-_{\text{BH}} \oplus D^+_{\text{BH}}$ with the dense domain $\mathcal{D}(D^-_{\text{BH}}) \oplus \mathcal{D}(D^+_{\text{BH}}) = \mathcal{P}_r[\mathcal{D}\left(D_{V_{\text{BH}},]-\infty,1]}\right) \oplus \mathcal{D}\left(D_{V_{\text{BH}},[1,+\infty[}\right)]$ using definitions (79), (88) and (89). Since

$$(D_{\text{BH}} \pm i)^{-1} - (D^-_{\text{BH}} \oplus D^+_{\text{BH}} \pm i)^{-1}$$

is of finite rank and thus trace class on L^2_{BH}, Birman-Kuroda theorem (see [23]) assures that

$$W^{\pm}\left(D_{\text{BH}}, D^-_{\text{BH}} \oplus D^+_{\text{BH}}\right)$$

exists on L^2_{BH} and

$$Ran\left(W^{\pm}\left(D_{\text{BH}}, D^-_{\text{BH}} \oplus D^+_{\text{BH}}\right)\right) = P_{ac}\left(D_{\text{BH}}\right)L^2_{\text{BH}}.$$

Therefore, the following wave operator

$$W^{\pm}\left(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}^-_{\text{BH}} \oplus \boldsymbol{D}^+_{\text{BH}}\right) = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\text{BH}}_{ln} W^{\pm}\left(D_{\text{BH}}, D^-_{\text{BH}} \oplus D^+_{\text{BH}}\right)\mathcal{R}^{\text{BH}}_{ln} \tag{94}$$

exists on $\boldsymbol{L}^2_{\text{BH}}$, and

$$Ran\left(W^{\pm}\left(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}^-_{\text{BH}} \oplus \boldsymbol{D}^+_{\text{BH}}\right)\right) = P_{ac}(\boldsymbol{D}_{\text{BH}})\boldsymbol{L}^2_{\text{BH}}. \tag{95}$$

Now, as

$$|r - r_0| \leq \mathcal{O}\left(e^{2\kappa_0 r_*}\right)\ r_* \to -\infty, \quad |r - r_+| \leq \mathcal{O}\left(e^{2\kappa_+ r_*}\right)\ r_* \to +\infty,$$

we compare respectively, the self-adjoint operators $\boldsymbol{D}^-_{\text{BH}}$ and $\boldsymbol{D}^-_{\leftarrow}$ on $L^2(]-\infty,1]_{r_*}\times S^2_\omega)^4$ with the dense domain $\mathcal{D}(\boldsymbol{D}^-_{\leftarrow})$, given by

$$\begin{aligned}
&\boldsymbol{D}^-_{\leftarrow} := \Gamma^1\partial_{r_*} - \frac{qQ}{r_0},\\
&\mathcal{D}(\boldsymbol{D}^-_{\leftarrow}) = \big\{\Psi \in L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4;\\
&\qquad\qquad \boldsymbol{D}^-_{\leftarrow}\Psi \in L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4,\ \gamma^1\Psi(1,.) = i\Psi(1,.)\big\},
\end{aligned}$$

and, the self-adjoint operators $\boldsymbol{D}^+_{\text{BH}}$ and $\boldsymbol{D}^+_{\Lambda,\rightarrow}$ on $L^2([1,+\infty[_{r_*}\times S^2_\omega)^4$ with the dense domain $\mathcal{D}(\boldsymbol{D}^+_{\Lambda,\rightarrow})$, given by

$$\begin{aligned}
&\boldsymbol{D}^+_{\Lambda,\rightarrow} := \Gamma^1\partial_{r_*} - \frac{qQ}{r_+},\\
&\mathcal{D}(\boldsymbol{D}^+_{\Lambda,\rightarrow}) = \big\{\Psi \in L^2([1,+\infty[_{r_*}\times S^2_\omega,\ dr_*d\omega)^4;\\
&\qquad\qquad \boldsymbol{D}^+_{\Lambda,\rightarrow}\Psi \in L^2([1,+\infty[_{r_*}\times S^2_\omega,\ dr_*d\omega)^4,\ -\gamma^1\Psi(1,.) = i\Psi(1,.)\Big\}.
\end{aligned}$$

We introduce $\mathcal{J}_r$ such that

$$\mathcal{J}_r : \Psi(r_*,\omega) \to \mathcal{J}_r(\Psi)(r_*,\omega) = r^{-1}F^{-1/4}(r)\Psi(r_*,\omega) \tag{96}$$

and we apply respectively lemma 4.11 in [19] to $W^\pm(\mathcal{J}_r^{-1}\boldsymbol{D}^+_{\text{BH}}\mathcal{J}_r, \boldsymbol{D}^+_{\Lambda,\rightarrow})$ on $L^2([1,+\infty[_{r_*}\times S^2_\omega,\ dr_*d\omega)^4$ and to $W^\pm(\mathcal{J}_r^{-1}\boldsymbol{D}^-_{\text{BH}}\mathcal{J}_r, \boldsymbol{D}^-_{\leftarrow})$ on $L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4$. Hence

$$W^\pm(\boldsymbol{D}^-_{\text{BH}}, \boldsymbol{D}^-_{\leftarrow}, \mathcal{J}_r) \quad \Big(resp.\quad W^\pm(\boldsymbol{D}^+_{\text{BH}}, \boldsymbol{D}^+_{\Lambda,\rightarrow}, \mathcal{J}_r)\Big) \tag{97}$$

exists on $L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4$ (resp. $L^2([1,+\infty[_{r_*}\times S^2_\omega,\ dr_*d\omega)^4$), and

$$Ran\left(W^\pm(\boldsymbol{D}^-_{\text{BH}}, \boldsymbol{D}^-_{\leftarrow}, \mathcal{J}_r)\right) = P_{ac}(\boldsymbol{D}^-_{\text{BH}})L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4 \tag{98}$$

$$\Big(resp.\quad Ran\Big(W^\pm(\boldsymbol{D}^+_{\text{BH}}, \boldsymbol{D}^+_{\Lambda,\rightarrow}, \mathcal{J}_r)\Big) = P_{ac}(\boldsymbol{D}^+_{\text{BH}})L^2([1,+\infty[_{r_*}\times S^2_\omega,\ dr_*d\omega)^4\Big). \tag{99}$$

We introduce the operators $\mathcal{J}_-^*$ and $\mathcal{J}_+^*$ respectively as the adjoint of

$$\mathcal{J}_- : \Psi \to \mathcal{J}_-\Psi = \begin{cases} \chi_-\Psi & r_* \le 1\\ 0 & r_* \ge 1\end{cases},\ \chi_- \in C^\infty(\mathbb{R}_{r_*}),\ \exists\, a,b,\ a<b<1,\ \chi_-(r_*) = \begin{cases} 1 & r_* < a\\ 0 & r_* > b\end{cases} \tag{100}$$

and

$$\mathcal{J}_+ : \Psi \to \mathcal{J}_+\Psi = \begin{cases} \chi_+\Psi & r_* \ge 1\\ 0 & r_* \le 1\end{cases},\ \chi_+ \in C^\infty(\mathbb{R}_{r_*}),\ \exists\, a,b,\ 1<a<b,\ \chi_+(r_*) = \begin{cases} 1 & r_* > b\\ 0 & r_* < a\end{cases}. \tag{101}$$

Since $\boldsymbol{D}^-_{\leftarrow}$ on $L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4$ and $\boldsymbol{D}_{\leftarrow}$ on $\boldsymbol{L}^{2\pm}_{\leftarrow}$ have spherical symmetry, we use lemma 6.1 which gives the explicit calculation of the unitary group generated by these self-adjoint operators. Hence, for all $\Psi_0 \in C_0^\infty(]-\infty,1]_{r_*}\times S^2_\omega)^4$ and since $\partial_{r_*}\chi_-$ is compactly supported and $supp(\chi_-^2-1)\subset[a,+\infty[$:

$$\begin{aligned}
&\left\|\left(\boldsymbol{D}_{\leftarrow}\mathcal{J}_- - \mathcal{J}_-\boldsymbol{D}^-_{\leftarrow}\right)e^{it\boldsymbol{D}^-_{\leftarrow}}\Psi_0\right\|_{\boldsymbol{L}^2_{\leftarrow}} = \left\|(\partial_{r_*}\chi_-)\,e^{it\boldsymbol{D}^-_{\leftarrow}}\Psi_0\right\|_{L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4} \in L^1(\mathbb{R}_t),\\
&\left\|\left(\mathcal{J}_-^*\mathcal{J}_- - 1\right)e^{it\boldsymbol{D}^-_{\leftarrow}}\Psi_0\right\|_{L^2(]-\infty,1]_{r_*}\times S^2_\omega,\ dr_*d\omega)^4} \to 0,\ t\to\pm\infty.
\end{aligned}$$

Therefore, by a standard density argument, the wave operator $W^{\pm}(\boldsymbol{D}_{\leftarrow}, \boldsymbol{D}_{\leftarrow}^{-}, \mathcal{J}_{-})$ exists and is an isometry on $L^2(]-\infty,1]_{r_*} \times S^2_{\omega},\ dr_* d\omega)^4$. Moreover, if we take $\Psi \in \boldsymbol{L}^{2\pm}_{\leftarrow} \cap C_0^{\infty}(\mathbb{R}_{r_*} \times S^2_{\omega})^4$ such that, for real $R>0$, $\text{supp}\Psi_0^{\pm} \subset [R+1, -R+1]$, we obtain for $\pm T \gtrless \pm R$:

$$\mathcal{J}_{-}^{*} e^{iT\boldsymbol{D}_{\leftarrow}} \Psi_0^{\pm} = e^{-it\boldsymbol{D}_{\leftarrow}^{-}} \mathcal{J}_{-}^{*} e^{i(T+t)\boldsymbol{D}_{\leftarrow}} \Psi_0^{\pm} \quad \forall t \in \mathbb{R},$$
$$\left\| \left(\mathcal{J}_{-}\mathcal{J}_{-}^{*} - 1\right) e^{it\boldsymbol{D}_{\leftarrow}} \Psi_0^{\pm} \right\|_{\boldsymbol{L}^2_{\leftarrow}} \to 0,\ t \to \pm\infty,$$

since $supp(\chi_{-}^2 - 1) \subset [a, +\infty[$. Therefore, by density, the following wave operator

$$W^{\pm}(\boldsymbol{D}_{\leftarrow}^{-}, \boldsymbol{D}_{\leftarrow}, \mathcal{J}_{-}^{*}) \tag{102}$$

exists on $\boldsymbol{L}^{2\pm}_{\leftarrow}$, and

$$Ran\left(W^{\pm}(\boldsymbol{D}_{\leftarrow}^{-}, \boldsymbol{D}_{\leftarrow}, \mathcal{J}_{-}^{*})\right) = P_{ac}(\boldsymbol{D}_{\leftarrow}^{-}) L^2(]-\infty,1]_{r_*} \times S^2_{\omega},\ dr_* d\omega)^4. \tag{103}$$

Using again the lemma 6.1, which gives the explicit calculation of the unitary group generated by the self-adjoint operator $\boldsymbol{D}_{\leftarrow}^{+}$ on $L^2([1,+\infty[_{r_*} \times S^2_{\omega},\ dr_* d\omega)^4$, as above and in the same way, we deduce that the wave operator :

$$W^{\pm}(\boldsymbol{D}^{+}_{\Lambda,\rightarrow}, \boldsymbol{D}_{\Lambda,\rightarrow}, \mathcal{J}_{+}^{*}) \tag{104}$$

exists on $\boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}$, and

$$Ran\left(W^{\pm}(\boldsymbol{D}^{+}_{\Lambda,\rightarrow}, \boldsymbol{D}_{\Lambda,\rightarrow}, \mathcal{J}_{+}^{*})\right) = P_{ac}(\boldsymbol{D}^{+}_{\Lambda,\rightarrow}) L^2([1,+\infty[_{r_*} \times S^2_{\omega},\ dr_* d\omega)^4. \tag{105}$$

We define the operators :

$$\mathcal{J}_{\leftarrow}^{-} : L^2(]-\infty,1]_{r_*} \times S^2_{\omega}, r^2 F^{1/2}(r) dr_* d\omega)^4 \to \boldsymbol{L}^2_{\text{BH}}; \quad \Psi \mapsto \mathcal{J}_{\leftarrow}^{-}\Psi = \begin{cases} \Psi & r_* \leq 1 \\ 0 & r_* \geq 1 \end{cases} \tag{106}$$

$$\mathcal{J}_{\rightarrow}^{+} : L^2([1,+\infty[_{r_*} \times S^2_{\omega}, r^2 F^{1/2}(r) dr_* d\omega)^4 \to \boldsymbol{L}^2_{\text{BH}}; \quad \Psi \mapsto \mathcal{J}_{\rightarrow}^{+}\Psi = \begin{cases} \Psi & r_* \geq 1 \\ 0 & r_* \leq 1 \end{cases}, \tag{107}$$

and the chain rule applied to (94)(95), (97)(99), (102)(103), (104)(105) assures that

$$W^{\pm}(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}_{\leftarrow}, \mathcal{J}_{\leftarrow}^{-}\mathcal{J}_r\mathcal{J}_{-}^{*}) \oplus W^{\pm}(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}_{\Lambda,\rightarrow}, \mathcal{J}_{\rightarrow}^{+}\mathcal{J}_r\mathcal{J}_{+}^{*})$$

exists on $\boldsymbol{L}^{2\pm}_{\leftarrow} \oplus \boldsymbol{L}^{2\mp}_{\Lambda,\rightarrow}$. By proposition 6.3, the spectrum of $\boldsymbol{D}_{\text{BH}}$ is purely absolutely continuous when $\Lambda > 0$. Hence

$$Ran\left(W^{\pm}(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}_{\leftarrow}, \mathcal{J}_{\leftarrow}^{-}\mathcal{J}_r\mathcal{J}_{-}^{*}) \oplus W^{\pm}(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}_{\Lambda,\rightarrow}, \mathcal{J}_{\rightarrow}^{+}\mathcal{J}_r\mathcal{J}_{+}^{*})\right) = \boldsymbol{L}^2_{\text{BH}}.$$

Finally

$$W^{\pm}_{\leftarrow} \oplus W^{\pm}_{\rightarrow} = W^{\pm}(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}_{\leftarrow}, \mathcal{J}_{\leftarrow}^{-}\mathcal{J}_r\mathcal{J}_{-}^{*}) \oplus W^{\pm}(\boldsymbol{D}_{\text{BH}}, \boldsymbol{D}_{\Lambda,\rightarrow}, \mathcal{J}_{\rightarrow}^{+}\mathcal{J}_r\mathcal{J}_{+}^{*}) \quad in \quad \boldsymbol{L}^2_{\text{BH}}$$

because for all $\Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow} = \boldsymbol{L}^{2\pm}_{\Lambda,\rightarrow}$

$$\left\| \left\{\mathcal{J}_{\leftarrow} - \mathcal{J}_{\leftarrow}^{-}\mathcal{J}_r\mathcal{J}_{-}^{*}\right\} e^{it\boldsymbol{D}_{\leftarrow}} \Psi^{\pm} \right\| \leq \left\| \left\{\chi_{\leftarrow} - \chi_{-}\right\} e^{it\boldsymbol{D}_{\leftarrow}} \Psi^{\pm} \right\|_{L^2(]-\infty,1]_{r_*} \times S^2_{\omega},\ dr_* d\omega)^4} \to 0,\ t \to \pm\infty, \tag{108}$$

$$\left\| \left\{\mathcal{J}_{\Lambda,\rightarrow} - \mathcal{J}_{\rightarrow}^{+}\mathcal{J}_r\mathcal{J}_{+}^{*}\right\} e^{it\boldsymbol{D}_{\Lambda,\rightarrow}} \Psi^{\mp} \right\| \leq \left\| \left\{\chi_{\rightarrow} - \chi_{+}\right\} e^{it\boldsymbol{D}_{\Lambda,\rightarrow}} \Psi^{\mp} \right\|_{L^2([1,+\infty[_{r_*} \times S^2_{\omega},\ dr_* d\omega)^4} \to 0,\ t \to \pm\infty. \tag{109}$$

Indeed, taking $\Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow} \cap C_0^{\infty}(\mathbb{R}_{r_*} \times S^2_{\omega})^4 = \boldsymbol{L}^{2\pm}_{\Lambda,\rightarrow} \cap C_0^{\infty}(\mathbb{R}_{r_*} \times S^2_{\omega})^4$ we have

$$e^{it\boldsymbol{D}_{\leftarrow}} \Psi^{\pm}(r_*) = e^{itqQr_0^{-1}} \Psi^{\pm}(r_* \pm t), \quad e^{it\boldsymbol{D}_{\Lambda,\rightarrow}} \Psi^{\mp}(r_*) = e^{itqQr_+^{-1}} \Psi^{\mp}(r_* \pm t)$$

and, since $\chi_{\leftarrow} - \chi_{-}$ and $\chi_{\rightarrow} - \chi_{+}$ are compactly supported, by density we obtain the limits (108) and (109) for all $\Psi^{\pm} \in \boldsymbol{L}^{2\pm}_{\leftarrow} = \boldsymbol{L}^{2\pm}_{\Lambda,\rightarrow}$. ■

6.3 Sharp estimate of $\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\boldsymbol{U}(0,T)$: proof of theorem 5.2

We briefly describe the steps of the proof. First, we take advantage of the spherical invariance to reduce our study to a one dimensional problem. Since with (74), (75) and (87) we have

$$\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\boldsymbol{U}(0,T) = e^{-iT\delta} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln}\mathbf{1}_{[\delta,+\infty[}(D_{V_{l,\nu},0})U_{V_{l,\nu}}(0,T)\mathcal{R}^{\nu}_{ln}, \quad \delta := \frac{qQ}{r_0},$$

it is sufficient to study the propagator $\mathbf{1}_{[\delta,+\infty[}(D_{V_{l,\nu},0})U_{V_{l,\nu}}(0,T)$. Now, to simplify the notations, we forget subscripts ln and ν. We choose $\mathcal{J} \in C^{\infty}(\mathbb{R}_{r_*})$ satisfying

$$\exists\, a,b \in \mathbb{R},\ 0 < a < b < 1 \quad \mathcal{J}(r_*) = \left\{ \begin{array}{ll} 1 & r_* < a \\ 0 & r_* > b \end{array} \right. \tag{110}$$

and split in two parts our investigation:

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0})U_V(0,T) = \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T) + \mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T). \tag{111}$$

Far from the star, we treat the term $\mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T)$ using theorem 4.1 on the scattering by the eternal black-hole. Indeed, we have:

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T) = \mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_{V,\mathbb{R}}(-T),$$

seeing that

$$\boldsymbol{U}(t) = e^{-it\frac{qQ}{r_0}} \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln}U_{V_{l,\nu},\mathbb{R}}(t)\mathcal{R}^{\nu}_{ln}$$

where $U_{V_{l,\nu},\mathbb{R}}$ is defined by proposition 6.2. Near the star, with Φ^0 in *ad-hoc* dense subspace on $L^2_{\mathbb{R}}$, we note that $\mathcal{J}U_V(0,T)\Phi^0$ is given by $\mathcal{J}\Phi_{V,g_T}(0,r_*)$. The function $\Phi_{V,g_T}(t,r_*)$ is the only solution of the mixed characteristic problem (81) and (82) with initial data $g_T(t)$ specified on the characteristic sub-manifold $\Gamma := \{(t,r_*) \in \mathbb{R}_t \times [z(t),+\infty[;\ r_* = 1-t\}$ such that

$$g_T(t) := {}^t(0, [U_V(t,T)\Phi^0(1-t)]_2, [U_V(t,T)\Phi^0(1-t)]_3, 0), \quad \exists\, t_g > 0 :\ t > t_g \Rightarrow g_T(t) = 0.$$

Concurrently, in L^2 norm, we prove that

$$g_T(t) \sim g^{\frac{T}{2}}(t) := (W^-_{0,\mathbb{R}}\Phi^0)(1-2t-T), \quad T \to +\infty,$$

where the wave operator $W^-_{0,\mathbb{R}}$ is defined in lemma 6.3, seeing that

$$\mathcal{P}_r\left(\boldsymbol{W}^-_{\leftarrow}\right)^* = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^{\nu}_{ln}W^-_{0,\mathbb{R}}\mathcal{R}^{\nu}_{ln}. \tag{112}$$

Then, in L^2_0 norm we obtain

$$\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)\Phi^0 \sim \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}\Phi_{V,g^{T/2}} \sim \mathbf{1}_{[0,+\infty[}(D_{0,0})\mathcal{J}\Phi_{0,g^{T/2}}, \quad T \to +\infty.$$

The last term entails asymptotically an explicit calculation which leads to a term of KMS-type depending on $W^-_{0,\mathbb{R}}$. This proof using the characteristic problem allows us to easily introduce the wave operator $W^-_{0,\mathbb{R}}$. This operator is connected with the curvature of the space-time at the vicinity of the eternal black-hole horizon. To finish, we prove that the two terms on the right hand side in (111) are asymptotically orthogonal as $T \to +\infty$.

6.3.1 Preliminary estimate for $\mathbf{1}_{[\delta,+\infty[}(D_{V,0})U_V(0,T)$

In this part, we use the notations introduce by formulas (74) (75), (88) (89) and propositions (6.1)(6.2). Therefore, we note that

$$D_{V,0} = D_{V,[z(0),+\infty[}, \quad L^2_0 = L^2([z(0),+\infty[_{r_*},\ dr_*)^4. \tag{113}$$

Since $D_{\text{BH}} = e^{-i\frac{\nu}{2}\gamma^5}\mathcal{P}_r D_{V,\mathbb{R}}\, e^{i\frac{\nu}{2}\gamma^5}\mathcal{P}_r^{-1} - \frac{qQ}{r_0}$, then, thanks to proposition 6.3, we have $\sigma(D_{V,\mathbb{R}}) = \sigma_{ac}(D_{V,\mathbb{R}})$ and therefore we deduce the following lemma of local energy decay:

Lemma 6.2
If $\Lambda \geq 0$, then

$$\lim_{t\to\pm\infty} \left\| f\ U_{V,\mathbb{R}}(t)\Phi \right\| = 0,$$

with $f \in C^0(\mathbb{R}, \mathcal{M}_4(\mathbb{C}))$ and $\lim_{r_*\to\pm\infty} |f(r_*)| = 0$.

Proof:
We consider the dense subspace $\mathcal{L}_d(D_{V,\mathbb{R}})$ in $L^2_{\mathbb{R}}$ such that

$$\mathcal{L}_d(D_{V,\mathbb{R}}) = \left\{\Phi \in L^2_{\mathbb{R}};\ \ B \subset \mathbb{R},\ |B| < +\infty,\ \mathbf{1}_B(D_{V,\mathbb{R}})\Phi = \Phi\right\}.$$

As $\sigma(D_{V,\mathbb{R}}) = \sigma_{ac}(D_{V,\mathbb{R}})$, we have $U_{V,\mathbb{R}}(t) \rightharpoonup 0,\ t \to \pm\infty$. Then for all $\Phi \in \mathcal{L}_d(D_{V,\mathbb{R}})$

$$\lim_{t\to\pm\infty} \left\| f\ \mathbf{1}_B(D_{V,\mathbb{R}})U_{V,\mathbb{R}}(t)\Phi \right\| = 0,$$

because $f\mathbf{1}_B(D_{V,\mathbb{R}})$ is compact on $L^2_{\mathbb{R}}$ following proposition B.7.1 in [7]. Hence, by a density argument, the limit is proved for $\Phi \in L^2_{\mathbb{R}}$. ■

We choose a cut-off function $\chi \in C^\infty(\mathbb{R}_{r_*})$, such that

$$\exists\ a, b \in \mathbb{R},\ \ -\infty < a < b < +\infty \quad \chi(r_*) = \begin{cases} 1 & r_* < a \\ 0 & r_* > b \end{cases}, \tag{114}$$

and the subspaces $L^{2+}_{\mathbb{R}}$, $L^{2-}_{\mathbb{R}}$ of $L^2_{\mathbb{R}}$, satisfying :

$$L^{2+}_{\mathbb{R}} = \left\{\Phi \in L^2_{\mathbb{R}};\ \Phi_2 \equiv \Phi_3 \equiv 0\right\}, \quad L^{2-}_{\mathbb{R}} = \left\{\Phi \in L^2_{\mathbb{R}};\ \Phi_1 \equiv \Phi_4 \equiv 0\right\}.$$

Therefore, we state the lemma:

Lemma 6.3
The wave operators

$$W^{\pm}_{0,\mathbb{R}} = s - \lim_{t\to\pm\infty} U_{0,\mathbb{R}}(-t)\chi U_{V,\mathbb{R}}(t), \quad in \quad L^2_{\mathbb{R}}$$
$$W^{\pm}_{V,[z(0),+\infty[} = s - \lim_{t\to\pm\infty} U_{V,[z(0),+\infty[}(-t)(1-\chi)U_{V,\mathbb{R}}(t) \quad in \quad L^2_0 = L^2([z(0),+\infty[_{r_*},\ dr_*)^4$$

exist and are independent of χ satisfying (114). Moreover

$$Ran\left(W^{\pm}_{0,\mathbb{R}}\right) = L^{2\pm}_{\mathbb{R}}, \quad Ran\left(W^{\pm}_{V,[z(0),+\infty[}\right) = P_{ac}\left(D_{V,[z(0),+\infty[}\right) L^2_0 \tag{115}$$

where $P_{ac}\left(D_{V,[z(0),+\infty[}\right)$ is the projector on the absolutely continuous subspace of $D_{V,[z(0),+\infty[}$, and for $f \in H^1_{\mathbb{R}}$

$$\lim_{t\to-\infty} \left\| U_{0,\mathbb{R}}(t)\left(W^-_{0,\mathbb{R}} f\right) - \chi U_{V,\mathbb{R}}(t) f \right\|_{H^1_{\mathbb{R}}} = 0. \tag{116}$$

Proof:
For the wave operator $W^{\pm}_{0,\mathbb{R}}$, the existence and property (115) are contained in theorem 4.1, since (112) exists and is an isometry from $\boldsymbol{L}^2_{\text{BH}}$ onto $\boldsymbol{L}^2_{\text{BH}} := \mathcal{P}_r \boldsymbol{L}^2_{\leftarrow}$. For $W^{\pm}_{V,[z(0),+\infty[}$, we note that

$$\left(D_{V,]-\infty,z(0)]} \oplus D_{V,[z(0),+\infty[} \pm i\right)^{-1} - \left(D_{V,\mathbb{R}} \pm i\right)^{-1}$$

is of finite rank. Then, with the notations (93) introduced in the proof of theorem 4.1, we obtain by the Birman-Kuroda theorem the existence on $L^2_{\mathbb{R}}$ of the wave operator

$$W^{\pm}\left(D_{0,]-\infty,z(0)]}, D_{V,\mathbb{R}}, \mathcal{J}_1\right) \oplus W^{\pm}\left(D_{V,[z(0),+\infty[}, D_{V,\mathbb{R}}, \mathcal{J}_2\right) = W^{\pm}\left(D_{0,]-\infty,z(0)]} \oplus D_{V,[z(0),+\infty[}, D_{V,\mathbb{R}}\right),$$

where

$$\mathcal{J}_1 : \Phi \in L^2_{\mathbb{R}} \mapsto \mathcal{J}_1 \Phi = \Phi_{|]-\infty,z(0)]} \in L^2(]-\infty, z(0)]_{r_*},\ dr_*)^4,$$
$$\mathcal{J}_2 : \Phi \in L^2_{\mathbb{R}} \mapsto \mathcal{J}_2 \Phi = \Phi_{|[z(0),+\infty[} \in L^2([z(0), +\infty[_{r_*},\ dr_*)^4 = L^2_0,$$

with the property

$$Ran\left(W^{\pm}\left(D_{V,]-\infty,z(0)]} \oplus D_{V,[z(0),+\infty[}, D_{V,\mathbb{R}}\right)\right) = \left[P_{ac}\left(D_{V,]-\infty,z(0)]}\right) \oplus P_{ac}\left(D_{V,[z(0),+\infty[}\right)\right] L^2_{\mathbb{R}}.$$

Now, we must show the equality:

$$W^{\pm}_{V,[z(0),+\infty[} = W^{\pm}\left(D_{V,[z(0),+\infty[}, D_{V,\mathbb{R}}, \mathcal{J}_2\right). \tag{117}$$

It arises from lemma 6.2. Indeed, for all $\Phi \in L^2_{\mathbb{R}}$, we have

$$\left\|[\mathcal{J}_2 - (1-\chi)] U_{V,\mathbb{R}}(t)\Phi\right\|_{L^2_0} \leq \left\|\mathbf{1}_{[z(0),+\infty[}\chi U_{V,\mathbb{R}}(t)\Phi\right\| \to 0,\ t \to \pm\infty,$$

because $\lim_{|r_*|\to+\infty} \mathbf{1}_{[z(0),+\infty[}\chi = 0$. Now we prove property (116). Since wave operator $W^{-}_{0,\mathbb{R}}$ exists, then

$$W^{-}_{0,\mathbb{R}} D_{V,\mathbb{R}} = D_{0,\mathbb{R}} W^{-}_{0,\mathbb{R}}.$$

Given $f \in H^1_{\mathbb{R}} = \mathcal{D}(D_{V,\mathbb{R}})$, then there exists $\Phi \in L^2_{\mathbb{R}}$ such that $\Phi = D_{V,\mathbb{R}} f$. Therefore, with the previous formula

$$\begin{aligned}
\left\|U_{0,\mathbb{R}}(t)\left(W^{-}_{0,\mathbb{R}} f\right) - \chi U_{V,\mathbb{R}}(t) f\right\|_{H^1_{\mathbb{R}}} &\leq \left\|D_{0,\mathbb{R}} U_{0,\mathbb{R}}(t)\left(W^{-}_{0,\mathbb{R}} f\right) - \chi D_{V,\mathbb{R}} U_{V,\mathbb{R}}(t) f\right\| \\
&\quad + \left\|\left\{\chi V + [\chi, D_{0,\mathbb{R}}]\right\} U_{V,\mathbb{R}}(t) f\right\| + \left\|U_{0,\mathbb{R}}(t)\left(W^{-}_{0,\mathbb{R}} f\right) - \chi U_{V,\mathbb{R}}(t) f\right\|, \\
&= \left\|U_{0,\mathbb{R}}(t)\left(W^{-}_{0,\mathbb{R}} \Phi\right) - \chi U_{V,\mathbb{R}}(t)\Phi\right\| \\
&\quad + \left\|\left\{\chi V + [\chi, D_{0,\mathbb{R}}]\right\} U_{V,\mathbb{R}}(t) f\right\| + \left\|U_{0,\mathbb{R}}(t)\left(W^{-}_{0,\mathbb{R}} f\right) - \chi U_{V,\mathbb{R}}(t) f\right\|.
\end{aligned}$$

The first and the third norm on the right hand side are treated by the previous scattering results for $W^{-}_{0,\mathbb{R}}$, and the second using lemma 6.2, since $\lim_{r_*\to\pm\infty}(|\chi V| + |[\chi, D_{0,\mathbb{R}}]|) = 0$. ■
Now, we solve the characteristic Cauchy problem

Lemma 6.4
For any $g := {}^t(0, g_2, g_3, 0) \in H^1_{\mathbb{R}}$, such that $t > t_g \Rightarrow g(t) = 0$, then there exists an unique solution Φ of

$$\partial_t \Phi = i\Gamma^1 \partial_{r_*}\Phi + iV\Phi, \quad t \in \mathbb{R}, \quad z(t) < r_* < -t+1, \tag{118}$$
$$\Phi_4(t, z(t)) = Z(t)\Phi_2(t, z(t)), \quad \Phi_1(t, z(t)) = -Z(t)\Phi_3(t, z(t)), \quad t \in \mathbb{R}, \tag{119}$$
$$(0, \Phi_2, \Phi_3, 0)(t, -t+1) = g(t), \quad t \in \mathbb{R}, \tag{120}$$
$$t > t_g,\ r_* \in [z(t), -t+1] \Rightarrow \Phi(t, r_*) = 0, \tag{121}$$

with $\widetilde{\Phi} \in C^1(\mathbb{R}_t, L^2_{\mathbb{R}}) \cap C^0(\mathbb{R}_t, H^1_{\mathbb{R}})$ such that

$$t \in \mathbb{R},\ r_* \in [z(t), -t+1] \Rightarrow \Phi(t, r_*) = \widetilde{\Phi}(t, r_*). \tag{122}$$

Proof:
We prove the uniqueness. Given Φ a solution of the problem for $g \equiv 0$ such that $\widetilde{\Phi} \in C^1(\mathbb{R}_t, L^2_{\mathbb{R}}) \cap C^0(\mathbb{R}_t, H^1_{\mathbb{R}})$ and $z(t) < r_* \Rightarrow \widetilde{\Phi}(t, r_*) = \Phi(t, r_*)$. We have for $t \in \mathbb{R}$:

$$\begin{aligned}
&\frac{d}{dt} \int_{z(t)}^{-t+1} |\Phi|^2 (t, r_*) dr_* \\
&\quad = -|\Phi|^2 (t, -t+1) - \dot{z}(t) |\Phi|^2 (t, z(t)) + 2 \int_{z(t)}^{-t+1} \Re < \partial_t \Phi, \Phi >_{\mathbb{C}^4} (t, r_*) dr_*, \\
&\quad = -2 |\Phi_2|^2 (t, -t+1) - 2 |\Phi_3|^2 (t, -t+1) \\
&\qquad\qquad + 2 \int_{z(t)}^{-t+1} \Re < \partial_t \Phi - i\Gamma^1 \partial_{r_*} \Psi - iV\Phi, \Phi >_{\mathbb{C}^4} (t, r_*) dr_*.
\end{aligned}$$

Since $\Phi(t, r_*)$ satisfies equation (118), then

$$\frac{d}{dt} \int_{z(t)}^{-t+1} |\Phi|^2 (t, r_*) dr_* = -2 |\Phi_2|^2 (t, -t+1) - 2 |\Phi_3|^2 (t, -t+1). \tag{123}$$

Integrating (123) on $[t, T]$, $T > t_g$ with respect to time, we obtain with (121),

$$\int_{z(t)}^{-t+1} |\Phi|^2 (t, r_*) dr_* = 2 \int_t^{+\infty} |g|^2(\tau) d\tau \leq 2\|g\|^2. \tag{124}$$

Therefore, since $g \equiv 0$ then $\Phi \equiv 0$.
Now, we prove the existence of the solution for a regular initial data $g = (0, g_2, g_3, 0) \in C^1_0(\mathbb{R})^4$. First, we solve the following characteristic problem:

$$\partial_t f_V = i\Gamma^1 \partial_{r_*} f_V + iV f_V, \quad t \in \mathbb{R}, \quad r_* > -t+1, \tag{125}$$

$$f_V(t, -t+1) = g(t), \quad t \in \mathbb{R}, \tag{126}$$

$$t \in]1 - r_*, r_* + a[\ \Rightarrow f_V(t, r_*) = 0, \tag{127}$$

where

$$a = \inf \left[supp(g) \right].$$

The continuous solution f_V of (125), (126) and (127) is given by the continuous solution of the following equivalent integral problem:

$$f_V(t, r_*) = F(X = t + r_* - 1, T = t - r_* - a) = \left\{ \begin{array}{ll} g\left(\frac{T+a+1}{2}\right) + \mathcal{B}F(X, T) & X \geq 0, \ T > 0, \\ 0 & X \geq 0, \ T \leq 0, \end{array} \right. , \tag{128}$$

$$\mathcal{B}F(X, T) = \frac{i}{2} \begin{pmatrix} \int_0^T \left[V\left(\frac{X-\xi-a+1}{2}\right) F(X, \xi) \right]_1 d\xi \\ \int_0^X \left[V\left(\frac{\xi-T-a+1}{2}\right) F(\xi, T) \right]_2 d\xi \\ \int_0^X \left[V\left(\frac{\xi-T-a+1}{2}\right) F(\xi, T) \right]_3 d\xi \\ \int_0^T \left[V\left(\frac{X-\xi-a+1}{2}\right) F(X, \xi) \right]_4 d\xi \end{pmatrix}. \tag{129}$$

For $X \geq 0, \ T > 0$, putting

$$F^0(X, T) = g\left(\frac{T+a+1}{2}\right); \quad F^{n+1}(X, T) = \mathcal{B}F^n(X, T), \ n \geq 0,$$

and since, g and V are bounded, we have

$$\left| \mathcal{B}F^{n-1}(X, T) \right| \leq \|g\|_{L^\infty} \|V\|^n_{L^\infty} 6^n \frac{(X+T)^n}{n!}, \quad n \geq 1.$$

Then the Picard method, gives a unique solution $F(X,T) \in C^0([0,+\infty[_X \times \mathbb{R}_T)^4$ of (128) such that

$$F(X,T) = \sum_{n=0}^{+\infty} F^n(X,T), \quad |F(X,T)| \leq \|g\|_{L^\infty} \exp\left(6 \|V\|_{L^\infty} (|X| + |T|)\right).$$

Seeing that $(X \geq 0,\ T \leq 0) \Rightarrow F(X,T) = 0$, $V \in C^\infty(\mathbb{R}_{r_*}, \mathcal{M}_4(\mathbb{C}))$ and for $X, T \geq 0$

$$|\partial_X F^n(X,T)| + |\partial_Y F^n(X,T)| \leq 16 \|g\|_{L^\infty} \|V\|_{L^\infty}^n 12^{n-1} \frac{(X+T)^{n-1}}{(n-1)!}$$
$$+ 2 \|g\|_{L^\infty} \left\|V'\right\|_{L^\infty} \|V\|_{L^\infty}^{n-1} 12^n \frac{(X+T)^n}{n!} + \left\|g'\right\|_{L^\infty} \|V\|_{L^\infty}^n 6^n \frac{(X+T)^n}{n!}, \quad n \geq 1,$$

we have $F(X,T) \in C^1(\{(X,T) \in [0,+\infty[_X \times \mathbb{R}_T :\ 2t_g \geq X + T + a + 1\})^4$. Hence,

$$[\Phi_g]_H(.,.) = [U_V(.,t_g)\phi_V(t_g,.)]_H \in C^1(\mathbb{R}_t, L^2_{\mathbb{R}}) \cap C^0(\mathbb{R}_t, H^1_{\mathbb{R}}), \tag{130}$$

$$[\phi_V(t_g,.)]_H \in H^1_{\mathbb{R}}, \quad \phi_V(t_g,r_*) := \begin{cases} f_V(t_g,r_*) & r_* > -t_g + 1, \\ 0 & z(t_g) < r_* \leq -t_g + 1, \end{cases} \tag{131}$$

is a solution of (81), (82), (120) and (121) and in particular of (118), (119), (120) and (121) with $g \in C^1_0(\mathbb{R})$. Moreover we have

$$\frac{d}{dt} \int_{-t+1}^{+\infty} |f_V|^2 (t,r_*) dr_* = 2 |g|^2 (t),$$

and integrating this formula on $[-\infty, t_g]$ with respect to time, we obtain

$$\int_{-t_g+1}^{+\infty} |f_V|^2 (t_g,r_*) dr_* = 2 \|g\|^2 .$$

Thanks to (130), (131) and (86),

$$\sup_{t \in \mathbb{R}} \|[\Phi_g(t,.)]_L\|^2_{L^2_{\mathbb{R}}} = \sup_{t \in \mathbb{R}} \|\Phi_g(t,.)\|^2_t = 2 \|g\|^2 \tag{132}$$

and by density and continuity, we get the existence with $g \in H^1_{\mathbb{R}}$. ■

We introduce some notations: For $g \in L^2_{\mathbb{R}}$,

$$g^T(.) := g(. - T), \quad T \geq 0,$$

and following the previous lemma, when $g := {}^t(0, g_2, g_3, 0) \in H^1_{\mathbb{R}}$, $t > t_g \Rightarrow g(t) = 0$, we define the operator $G_V(g)$ such that

$$G_V(g)(r_*) := \mathcal{J}(r_*) \Phi_V(0, r_*), \quad r_* \in [z(0), 1], \tag{133}$$

with $\mathcal{J}$ as in (110) and $\Phi_V(0,r_*)$ the solution of (118), (119), (120) and (121). Moreover, by density and thanks to (124), formula (133) is well defined for $g \in L^2_{\mathbb{R}}$, $t > t_g \Rightarrow g(t) = 0$. Therefore, we prove the first important estimate:

Lemma 6.5
Given $g := {}^t(0, g_2, g_3, 0) \in L^2_{\mathbb{R}}$ such that $t > t_g \Rightarrow g(t) = 0$, then

$$\lim_{T \to +\infty} \left\| \mathbf{1}_{[0,+\infty[}(D_{0,0}) \left[G_0\left(g^T\right)\right]_L \right\|_0^2 = 2 < g, e^{\frac{2\pi}{\kappa_0} D_{0,\mathbb{R}}} \left(1 + e^{\frac{2\pi}{\kappa_0} D_{0,\mathbb{R}}}\right)^{-1} g >_{L^2_{\mathbb{R}}}, \tag{134}$$

and

$$\left[G_0\left(g^T\right)\right]_L \rightharpoonup 0, \quad T \to +\infty, \quad in \quad L^2_0. \tag{135}$$

Proof:
As the norm of (134) is uniformly bounded in T by (124), it is enough to obtain (134) for $g \in C_0^\infty(\mathbb{R})^4$ such that $supp(g) \subset [0,R]$, $R>0$ fixed. For $T > \frac{1-z(0)}{2}$, we have $G_0(g^T) \in [z(0),0[$ and thanks to lemma 6.1,

$$G_0(g^T)(r_*) = Z(\tau(r_*))\ {}^t(-g_3,0,0,g_2)^T\left(\tau(r_*)+\frac{1-r_*}{2}\right),$$

with τ and Z respectively defined by (8) and (76). We define spinor G^T, such that

$$G^T(r_*) := \frac{1}{\sqrt{-\kappa_0 r_*}}\,{}^t(-g_3,0,0,g_2)^T\left(-\frac{1}{2\kappa_0}\ln(-r_*)+\frac{1}{2\kappa_0}\ln(C_{\kappa_0})+\frac{1}{2}\right),\quad r_*<0, \tag{136}$$

with $supp(G^T) \subset]-\infty,0[$ and the real $C_k>0$ as in (8). In the first time, we remark that, for $f \in L^2_{\mathbb{R}}$,

$$< [G^T]_L, f >_{L^2_{\mathbb{R}}} \to 0, \quad T\to+\infty. \tag{137}$$

Indeed, for $f \in C_0^\infty(\mathbb{R})^4$, we have

$$\begin{aligned}\left|< [G^T]_L, f >_{L^2_{\mathbb{R}}}\right| &\leq \|f\|_{L^\infty(\mathbb{R})^4}\int_{\mathbb{R}}\left|[G^T]_L\right|(r_*)dr_* = \|f\|_{L^\infty(\mathbb{R})^4}\int_{-\infty}^0\left|G^T\right|(r_*)dr_*\\ &\leq \sqrt{\kappa_0 C_{\kappa_0}e^{-2\kappa_0 T+\kappa_0}}\|f\|_{L^\infty(\mathbb{R})^4}\int_{\mathbb{R}}e^{-\kappa_0 y}|g|(y)dy \to 0, \quad T\to+\infty.\end{aligned}$$

We obtain (137) by density and using the inequality $\|[G^T]_L\| \leq \|g\|$. Moreover, for $T > \frac{1-z(0)}{2}$, we have,

$$\left\|[G^T]_L - \left[G_0(g^T)\right]_L\right\|_0^2 = \int_{z(0)}^0\left|G^T(r_*) - G_0(g^T)\right|^2 dr_*.$$

We remark that: $Z(\tau(r_*)) \in C^0([z(0),0[)$ and

$$\lim_{r_*\to 0^-} h(r_*) = 1, \quad h(r_*) := \sqrt{-\kappa_0 r_*}Z(\tau(r_*)). \tag{138}$$

Indeed, thanks to (8) and (9), (138) entails that

$$h(r_*) = \sqrt{-\kappa_0 r_*}\sqrt{\frac{1-\dot z(\tau(r_*))}{1+\dot z(\tau(r_*))}} = \sqrt{\frac{-2\kappa_0 r_* + \mathcal{O}'({r_*}^2)}{-2\kappa_0 r_* + \mathcal{O}({r_*}^2)}}, \quad -1<\dot z(\tau(r_*))\leq 0, \quad r_*\in[z(0),0[.$$

Therefore, using (8) and putting $y(r_*) = -\frac{1}{2\kappa_0}\ln(-r_*)+\frac{1}{2\kappa_0}\ln(C_{\kappa_0})+\frac{1}{2}-T$,

$$\begin{aligned}&\left\|[G^T]_L - \left[G_0(g^T)\right]_L\right\|_0^2\\ &= 2\int_{-\frac{1}{2\kappa_0}\ln(-z(0))+\frac{1}{2\kappa_0}\ln(C_{\kappa_0})+\frac{1}{2}-T}^{+\infty}\left|g(y) - h\left(-C_{\kappa_0}e^{-2T\kappa_0-2y\kappa_0+\kappa_0}\right)g\left(y+\mathcal{O}(e^{-2T\kappa_0-2y\kappa_0+\kappa_0})\right)\right|^2 dy,\end{aligned}$$

and by Lebesgue theorem, we obtain:

$$\left\|\left[G^T\right]_L - \left[G_0(g^T)\right]_L\right\|_0^2 \to 0, \quad T\to+\infty. \tag{139}$$

With (137), this last limit gives (135). Finally, for $T > -\frac{1}{2\kappa_0}\ln(-z(0)) + \frac{1}{2\kappa_0}\ln(C_{\kappa_0}) + \frac{1}{2}$, and denoting $\mathcal{F}$ the Fourier transform, we have

$$\begin{aligned}
\left\|\mathbf{1}_{[0,+\infty[}(D_{0,0})[G^T]_L\right\|_0^2 &= \frac{1}{2\pi}\int_0^{+\infty} \left|\mathcal{F}\left([G^T]_L\right)\right|^2(\xi)d\xi \qquad (140)\\
&= \frac{C_{\kappa_0}\kappa_0}{2\pi}\int_0^{+\infty}\left|\int_{\mathbb{R}} e^{iC_{\kappa_0}\xi e^{\kappa_0 y}} e^{\frac{\kappa_0}{2}y}\tilde{g}(y)dy\right|^2 d\xi, \quad \tilde{g}(y) = g(-y/2),\\
&=< \tilde{g}, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1}\tilde{g} >_{L^2_{\mathbb{R}}}, \quad \text{(lemma III.6 in [4])},\\
&= 2 < g, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} g >_{L^2_{\mathbb{R}}},
\end{aligned}$$

which implies, with (139), limit (134). ■

To prove the following estimate, we need a Gronwall type inequality:

Lemma 6.6

Given $J, E_1, E_2 \in C^0([a,b])$ and $t \in [a,b] \Rightarrow E_1(t),\ E_2(t) \geq 0$, such that

$$J(t) \leq E_2(t) + E_1(t)\int_a^t J(s)ds, \quad a \leq t \leq b, \qquad (141)$$

then

$$J(t) \leq E_2(t) + E_1(t)\exp\left(\int_a^t E_1(s)ds\right)\int_a^t E_2(s)ds, \quad a \leq t \leq b. \qquad (142)$$

Proof:

We put

$$R(s) = \exp\left(-\int_a^s E_1(\tau)d\tau\right)\int_a^s J(\tau)d\tau.$$

We differentiate $R(s)$ and using (141):

$$\begin{aligned}
\frac{d}{ds}R(s) &= J(s)\exp\left(-\int_a^s E_1(\tau)d\tau\right) - E_1(s)\exp\left(-\int_a^s E_1(\tau)d\tau\right)\int_a^s J(\tau)d\tau,\\
&\leq E_2(s)\exp\left(-\int_a^s E_1(\tau)\right).
\end{aligned}$$

As $R(a) = 0$, integrating the result on $[a,t]$, we obtain

$$R(t) \leq \int_a^t E_2(s)\exp\left(-\int_a^s E_1(\tau)d\tau\right)ds.$$

Since $s \in [a,t]$ and E_1 is non negative:

$$\exp\left(-\int_a^s E_1(\tau)d\tau\right) \leq 1.$$

Hence

$$\int_a^t J(s)ds \leq \exp\left(\int_a^t E_1(\tau)d\tau\right)\int_a^t E_2(s)ds,$$

and (142) follows. ■

Lemma 6.7
Given $g := {}^t(0, g_2, g_3, 0) \in L^2_{\mathbb{R}}$ *such that* $t > t_g \Rightarrow g(t) = 0$, *then*

$$\lim_{T\to+\infty} \left\| \left[G_0\left(g^T\right)\right]_L - \left[G_V\left(g^T\right)\right]_L \right\|_0^2 = 0, \tag{143}$$

and

$$\left[G_V\left(g^T\right)\right]_L \rightharpoonup 0, \quad T \to +\infty, \quad in \quad L^2_0. \tag{144}$$

Proof:
With (124), it is enough to obtain the result for $g \in C_0^\infty(\mathbb{R})^4$ such that $supp(g) \subset [0, R]$, $R > 0$ fixed. By lemma 6.4, formulas (130) and (131), for $r_* \in [z(0), 1]$, we have

$$\begin{aligned} &G_V\left(g^T\right)(r_*) = \mathcal{J}(r_*)\left[U_V(0, R+T)\phi_V(R+T, .)\right](r_*), \\ &\phi_V(R+T, r_*) = \left\{ \begin{array}{ll} f_V(R+T, r_*) & r_* > -R-T+1, \\ 0 & z(R+T) < r_* \leq -R-T+1. \end{array} \right. \end{aligned} \tag{145}$$

Now, for $r_* \in [z(0), 1]$, we write

$$\begin{aligned} \left[G_V\left(g^T\right) - G_0\left(g^T\right)\right](r_*) &= \mathcal{J}(r_*)\left[U_V(0, R+T)\phi_V(R+T, .) - U_0(0, R+T)\phi_0(R+T, .)\right](r_*), \\ &= \mathcal{J}(r_*)\left[U_V(0, R+T)\left\{\phi_V(R+T, .) - \phi_0(R+T, .)\right\}\right](r_*) \\ &\qquad - \mathcal{J}(r_*)\left[\left\{U_0(0, R+T) - U_V(0, R+T)\right\}\phi_0(R+T, .)\right](r_*), \\ &=: A_1 + A_2. \end{aligned}$$

We estimate A_1. First, with (145), we have

$$\begin{aligned} \|A_1\|_0^2 &\leq \int_{z(R+T)}^{+\infty} \left|\phi_V(R+T, r_*) - \phi_0(R+T, r_*)\right|^2 dr_*, \\ &= \int_{-R-T+1}^{+\infty} \left|f_V(R+T, r_*) - f_0(R+T, r_*)\right|^2 dr_*, \\ &=: J(R+T). \end{aligned}$$

But,

$$\begin{aligned} \frac{d}{dt}J(t) &= \left|f_V - f_0\right|^2(t, -t+1) + 2\Re\int_{-t+1}^{+\infty} < \partial_t\left(f_V - f_0\right)(t, r_*), \left(f_V - f_0\right)(t, r_*) >_{\mathbb{C}^4} dr_*, \\ &=: J_1 + 2\Re J_2. \end{aligned}$$

Since the solutions f_V and f_0 have the same characteristic data, $J_1 = 0$. On the other hand, with the help of equations satisfied by f_V and f_0, we have:

$$\begin{aligned} J_2 &= \int_{-t+1}^{+\infty} < i\Gamma^1\partial_{r_*}\left(f_V - f_0\right)(t, r_*) + iVf_V(t, r_*), \left(f_V - f_0\right)(t, r_*) >_{\mathbb{C}^4} dr_*, \\ &= -\int_{-t+1}^{+\infty} < \left(f_V - f_0\right)(t, r_*), i\Gamma^1\partial_{r_*}\left(f_V - f_0\right)(t, r_*) >_{\mathbb{C}^4} dr_* \\ &\qquad + \int_{-t+1}^{+\infty} < iVf_V(t, r_*), \left(f_V - f_0\right)(t, r_*) >_{\mathbb{C}^4} dr_*, \\ &= -\int_{-t+1}^{+\infty} < \left(f_V - f_0\right)(t, r_*), i\Gamma^1\partial_{r_*}\left(f_V - f_0\right)(t, r_*) + iVf_V(t, r_*) >_{\mathbb{C}^4} dr_* \\ &\qquad + 2\Re\int_{-t+1}^{+\infty} < iVf_V(t, r_*), \left(f_V - f_0\right)(t, r_*) >_{\mathbb{C}^4} dr_*, \\ &= -\bar{J}_2 + 2\Re\int_{-t+1}^{+\infty} < iVf_V(t, r_*), \left(f_V - f_0\right)(t, r_*) >_{\mathbb{C}^4} dr_*. \end{aligned}$$

Then

$$\frac{d}{dt}J(t) = 2\Re\int_{-t+1}^{+\infty} < V(r_*)f_V(t,r_*), f_V(t,r_*) - f_0(t,r_*) >_{\mathbb{C}^4} dr_*. \tag{146}$$

In lemma 6.4, we have proved that the solution $f_V(t,x)$ propagates at speed one . Therefore, for $t \in [T, T+R]$, we have

$$\begin{aligned} supp\left(g^T\right) \subset [T,T+R] &\Rightarrow supp\left(f_V(t,.)\right) \subset [-t+1, t-2T+1], \quad T,R>0, \\ &\Rightarrow J(0) = 0. \end{aligned} \tag{147}$$

Hence, integrating (146) on $[0, T+R]$, we obtain:

$$J(R+T) = 2\Re\int_0^{T+R}\int_{-t+1}^{+\infty} < V(r_*)f_V(t,r_*), f_V(t,r_*) - f_0(t,r_*) >_{\mathbb{C}^4} dr_* dt.$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned} J(R+T) &\leq 2\int_0^{T+R}\int_{-t+1}^{+\infty} |< V(r_*)f_V(t,r_*), f_V(t,r_*) - f_0(t,r_*) >| \, dr_* dt, \\ &\leq 2\int_0^{T+R}\left(\int_{-t+1}^{+\infty} |V(r_*)f_V(t,r_*)|^2 \, dr_*\right)^{1/2} J(t)^{1/2} dt. \end{aligned}$$

Thanks to the remark (147) and as $\sqrt{x} \leq x+1$ for $x \geq 0$, then we deduce that

$$\begin{aligned} &J(R+T) \leq E_2(T+R) + E_1(T+R)\int_0^{T+R} J(t)dt, \\ &E_1(t) := 4\|g\| \sup\left\{|V(x)|;\ x \leq -t+2R+1\right\},\ E_2(t) := tE_1(t). \end{aligned}$$

As $E_1, J \in C^0(\mathbb{R})$, by lemma 6.6, we have

$$J(T+R) \leq E_2(T+R) + E_1(T+R)\exp\left(\int_0^{T+R} E_1(s)ds\right)\int_0^{T+R} E_2(s)ds.$$

Since, $V(r_*)$ is exponentially decreasing as $r_* \to -\infty$, we get

$$\|A_1\|_0^2 \leq J(R+T) \to 0, \quad T \to +\infty. \tag{148}$$

To estimate A_2, we use the usual formula

$$\left\{U_0(0,R+T) - U_V(0,R+T)\right\}\phi_0(R+T,.) = -\int_0^{R+T} U_V(0,s)VU_0(s,R+T)\phi_0(R+T,.)ds.$$

Hence, we deduce with (86) that

$$\begin{aligned} \|A_2\|_0 &\leq \left\|\left\{U_0(0,R+T) - U_V(0,R+T)\right\}\phi_0(R+T,.)\right\|_0, \\ &\leq \int_0^{R+T} \|VU_0(s,R+T)\phi_0(R+T,.)\|_s \, ds. \end{aligned} \tag{149}$$

Now we defined the time τ_T, such that

$$z(\tau_T) - \tau_T = -2T+1.$$

Thanks to (6):

$$\tau_T = T - \frac{1}{2} + \mathcal{O}\left(e^{-2\kappa_0 T}\right), \quad T \to +\infty \tag{150}$$

and according lemma 6.1, we have also

$$s \in [0, \tau_T] \Rightarrow [U_0(s, R+T)\phi_0(R+T, .)]\,(r_*) = Z(\tau(r_*))\ {}^t(-g_3, 0, 0, g_2)^T \left(\tau(r_*) + \frac{1 - r_* - s}{2}\right) \tag{151}$$

and

$$s \in [0, \tau_T] \Rightarrow supp\,[U_0(s, R+T)\phi_0(R+T, .)] \subset \left[-s - \left|\mathcal{O}(e^{-2\kappa_0 T})\right|, -s\right]. \tag{152}$$

Indeed, for $s \in [0, \tau_T]$,

$$supp\,[U_0(s, R+T)\phi_0(R+T, .)] \subset [-s + 2\tau_T - 2T + 1, -s],$$

and with (150), (152) follows. Hence,

$$\begin{aligned} \|A_2\|_0 &\leq \int_0^{\tau_T} \|VU_0(s, R+T)\phi_0(R+T, .)\|_s\, ds + \int_{\tau_T}^{R+T} \|VU_0(s, R+T)\phi_0(R+T, .)\|_s\, ds, \\ &\leq A_{21} + A_{22}. \end{aligned}$$

First, we estimate A_{21}. With the help of (152) and (151), we have,

$$A_{21} \leq \int_0^{\tau_T} \sqrt{I(s)}ds.$$

where

$$I(s) := \int_{-s-\left|\mathcal{O}(e^{-2\kappa_0 T})\right|}^{-s} \left| V(r_*) Z(\tau(r_*))\ {}^t(-g_3, 0, 0, g_2)^T \left(\tau(r_*) + \frac{1 - r_* - s}{2}\right)\right|^2 dr_*. \tag{153}$$

Using (8) and putting $y(r_*) = -\frac{1}{2\kappa_0}\ln(-r_*) + \frac{1}{2\kappa_0}\ln(C_{\kappa_0}) + \frac{1-s}{2} - T$, we have

$$\begin{aligned} I(s) &\leq 2\|V\|_{L^\infty}^2 \int_{y\left(-s-\left|\mathcal{O}\left(e^{-2\kappa_0 T}\right)\right|\right)}^{y(-s)} h^2(r_*(y, s, T))\, |g\,(y + \mathcal{O}(r_*(y, s, T)))|^2\, dy, \\ &\leq C_z \|V\|_{L^\infty}^2 \|g\|_{L^\infty}^2 \left[\ln\left(s + \left|\mathcal{O}(e^{-2\kappa_0 T})\right|\right) - \ln(s)\right], \quad C_z > 0, \end{aligned}$$

with h defined in (138). First, for $x \geq 0$, $\log(x+1) \leq x$. Hence we obtain

$$\begin{aligned} \int_0^{\tau_T} \sqrt{I(s)}ds &\leq C_{z,V,g} \int_0^{\tau_T} \sqrt{\ln\left(s + |\mathcal{O}(e^{-2\kappa_0 T})|\right) - \ln(s)}ds \\ &\leq C_{z,V,g}\ \left|\mathcal{O}(e^{-2\kappa_0 T})\right| \int_{C(T)}^{+\infty} \frac{\sqrt{\log(x+1)}}{x^2}dx, \quad C(T) := \left|\tau_T^{-1}\mathcal{O}(e^{-2\kappa_0 T})\right|, \\ &\leq C_{z,V,g}\ \left|\mathcal{O}(e^{-2\kappa_0 T})\right| \left(2\int_{C(T)}^1 \frac{\sqrt{x}}{x^2}dx + \int_1^{+\infty} \frac{\sqrt{\log(x+1)}}{x^2}dx\right), \\ &\leq C_{z,V,g}\ \left(\sqrt{\left|\tau_T \mathcal{O}(e^{-2\kappa_0 T})\right|} + C\left|\mathcal{O}(e^{-2\kappa_0 T})\right|\right) \to 0,\ T \to +\infty. \end{aligned}$$

For $s \in [\tau_T, T+R]$ we have: $supp[U_0(s,R+T)\phi_0(R+T,.)] \subset [z(s), 2R+1-s]$. Hence, thanks to (150) and (132),

$$A_{22} \leq \int_{\tau_T}^{R+T} \|VU_0(s,R+T)\phi_0(R+T,.)\|_s \, ds$$
$$\leq 2\|g\| \int_{\tau_T}^{T+R} \sup\{|V(x)|;\ z(s) \leq x \leq 2R+1-s\}\ ds \to 0, \quad T \to +\infty.$$

Then, we obtain that

$$\|A_2\|_0 \to 0, \quad T \to +\infty. \tag{154}$$

Now, finally, with (154) and (148), we deduce that

$$\left\| \left[G_0\left(g^T\right)\right]_L - \left[G_V\left(g^T\right)\right]_L \right\|_0 \leq \|A_1\|_0 + \|A_2\|_0 \to 0,\ T \to +\infty.$$

Lastly, the above result with (135), entails (144). ■

Lemma 6.8
Given $g := {}^t(0, g_2, g_3, 0) \in L^2_{\mathbb{R}}$ *such that* $t > t_g \Rightarrow g(t) = 0$, *then*

$$\lim_{T\to+\infty} \left\| \mathbf{1}_{[\delta,+\infty[}(D_{V,0}) \left[G_V\left(g^T\right)\right]_L \right\|_0^2 = 2 < g, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}} \left(1 + e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} g >_{L^2_{\mathbb{R}}}, \tag{155}$$

with

$$\delta = \frac{qQ}{r_0}.$$

Proof:
First, we define V_∞ thanks to V such that

$$V_\infty := \delta I_{\mathbb{R}^4} + \varsigma A_\nu = \lim_{r_*\to+\infty} V(r_*), \quad \delta = \frac{qQ}{r_0},\ \varsigma = -m\sqrt{F(r_+)}, \tag{156}$$

where A_ν as in (76). If $\varsigma < 0$ ($\Lambda = 0$), thus by assumption $\nu \neq (2k+1)\pi$, $k \in \mathbb{Z}$ and from the proof of lemma III-7 in [4], we set that:

$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty, z(0)]} \oplus D_{\varsigma A_\nu, [z(0),+\infty[}\right) - \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu, \mathbb{R}}\right) \quad \text{is compact.} \tag{157}$$

For $g \in C_0^\infty(\mathbb{R})^4$ such that $supp(g) \subset [0, R]$, $R > 0$ fixed, and $T > -\frac{1}{2\kappa_0}\ln(-z(0)) + \frac{1}{2\kappa_0}\ln(C_{\kappa_0}) + \frac{1}{2}$, we have $supp\left(G^T\right) \subset]z(0), 0[$ which entails:

$$\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,]-\infty, z(0)]} \oplus D_{\varsigma A_\nu, [z(0),+\infty[}\right)\left[G^T\right]_L = 0 \oplus \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu, [z(0),+\infty[}\right)\left[G^T\right]_L,$$

where G^T is defined by (136). Since,

$$\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right) = \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,0}\right) = \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu, [z(0),+\infty[}\right) \tag{158}$$

and according to (137) and (157), we deduce that

$$\left\| \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu, [z(0),+\infty[}\right)\left[G^T\right]_L - \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu, \mathbb{R}}\right)\left[G^T\right]_L \right\| \to 0, \quad T \to +\infty. \tag{159}$$

Seeing that $D_{\varsigma A_\nu, \mathbb{R}}$ is the Dirac Hamiltonian, using the Fourier transform $\mathcal{F}$:

$$\mathcal{F}\mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu, \mathbb{R}}\right) = \left[\frac{1}{2} + \frac{1}{2\sqrt{\xi^2 + \varsigma^2}}\left(i\xi\Gamma^1 + \varsigma A_\nu\right)\right]\mathcal{F}.$$

We remark that

$$\left|\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 = 4\kappa_0 B(T)|\theta(B(T)\xi)|^2,$$
$$\theta(B(T)\xi) := \int_{\mathbb{R}} e^{-\kappa_0 y} e^{i\xi B(T)e^{-2\kappa_0 y}} g(y)dy,\ \ B(T) := C_{\kappa_0} e^{-2\kappa_0 T+\kappa_0}.$$

Hence, thanks to Lebesgue's theorem,

$$\begin{aligned}
&\left\|\mathbf{1}_{[0,+\infty[}\left(D_{0,\mathbb{R}}\right)\left[G^T\right]_L - \mathbf{1}_{[0,+\infty[}\left(D_{\varsigma A_\nu,\mathbb{R}}\right)\left[G^T\right]_L\right\|^2 \\
&= C_1\int_{\mathbb{R}}\left|\frac{i\xi}{|\xi|}\Gamma^1 - \frac{1}{\sqrt{\xi^2+\varsigma^2}}\left(i\xi\Gamma^1+\varsigma A_\nu\right)\right|^2\left|\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 d\xi, \quad C_1>0, \\
&\leq C_2\int_{\mathbb{R}}\left|\frac{i\eta}{|\eta|}\Gamma^1 - \frac{1}{\sqrt{\eta^2+B^2(T)\varsigma^2}}\left(i\eta\Gamma^1+B(T)\varsigma A_\nu\right)\right|^2|\theta(\eta)|^2 d\eta \ \to 0,\ T\to+\infty,\ C_2>0.
\end{aligned} \tag{160}$$

By (140), we have,

$$\left\|\mathbf{1}_{[0,+\infty[}\left(D_{0,\mathbb{R}}\right)\left[G^T\right]_L\right\|^2 = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\mathcal{F}\left(\left[G^T\right]_L\right)(\xi)\right|^2 d\xi = \left\|\mathbf{1}_{[0,+\infty[}\left(D_{0,0}\right)\left[G^T\right]_L\right\|^2. \tag{161}$$

As $\|[G^T]_L\| \leq \|g\|$, by density and using (158), (159) ,(160), (161) and (139), we obtain, for $g \in L^2_{\mathbb{R}}$

$$\left|\left\|\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|^2 - \left\|\mathbf{1}_{[0,+\infty[}\left(D_{0,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|^2\right| \to 0, \quad T\to+\infty. \tag{162}$$

If $\varsigma = 0$ $(\Lambda > 0)$, then we have clearly:

$$\left\|\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|^2 = \left\|\mathbf{1}_{[0,+\infty[}\left(D_{0,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|^2. \tag{163}$$

Moreover, from lemma III-10 in [4], we have

$$\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right) - \mathbf{1}_{[\delta,+\infty[}\left(D_{V,0}\right) \quad \text{is compact.} \tag{164}$$

Then, using respectively (143), (164)-(135), (162)-(163) and (134), we conclude that:

$$\begin{aligned}
\lim_{T\to+\infty}\left\|\mathbf{1}_{[\delta,+\infty[}\left(D_{V,0}\right)\left[G_V\left(g^T\right)\right]_L\right\| &= \lim_{T\to+\infty}\left\|\mathbf{1}_{[\delta,+\infty[}\left(D_{V,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|, \\
&= \lim_{T\to+\infty}\left\|\mathbf{1}_{[\delta,+\infty[}\left(D_{V_\infty,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|, \\
&= \lim_{T\to+\infty}\left\|\mathbf{1}_{[0,+\infty[}\left(D_{0,0}\right)\left[G_0\left(g^T\right)\right]_L\right\|, \\
&= 2 < g, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} g >_{L^2_{\mathbb{R}}}.
\end{aligned}$$

■

We defined a dense subspace $\mathcal{D}_{\mathrm{R}}$ of $L^2_{\mathbb{R}}$, such that,

$$\mathcal{D}_{\mathrm{R}} = \left\{\Phi \in H^1_{\mathbb{R}} :\ \exists R>0\ \ r_* \leq R \Rightarrow \Phi(r_*) = 0\right\}.$$

For $f \in \mathcal{D}_{\mathrm{R}}$, we put

$$g_{\mathrm{T}}(t) := {}^t(0, [U_V(t,T)f]_2\,, [U_V(t,T)f]_3\,, 0)(-t+1), \tag{165}$$
$$g(t) = \left(W^-_{0,\mathbb{R}} f\right)(-2t+1), \tag{166}$$

where $[x]_j$ is the jth component of $x \in \mathbb{C}^4$. Moreover

$$2t \geq T-R+1 \Rightarrow g_{\mathrm{T}}(t) = 0, \tag{167}$$
$$2t \geq -R+1 \Rightarrow g(t) = 0. \tag{168}$$

Lemma 6.9
Given $f \in \mathcal{D}_{R}$, with the definitions (165) and (166) we have

$$\int_0^{+\infty} \left| g_T(t) - g^{\frac{T}{2}}(t) \right|^2 dt \to 0, \quad T \to +\infty. \tag{169}$$

Proof:
We define Φ such that

$$\Phi(t, r_*) := U_{V,\mathbb{R}}(t) f(r_*).$$

Since $f \in \mathcal{D}_{\mathrm{R}}$, then,

$$|t| \leq R - r_* \ \Rightarrow \ \Phi(t, r_*) = \left(W^-_{0,\mathbb{R}} f\right)(r_* - t) = 0. \tag{170}$$

Then, using the notation of (165) and for $t + r_* \leq R$, we remark that $\Phi(t, r_*)$ as is solution of

$$\Phi(t, r_*) = \left(W^-_{0,\mathbb{R}} f\right)(r_* - t) + \int_{-\infty}^{r_*} \mathcal{A}\Phi(t, r_*, s) ds, \tag{171}$$

with

$$j = 1, 4 \ \Rightarrow \ [\mathcal{A}\Phi(t, r_*, s)]_j = -[iV(s)\Phi(r_* - s + t, s)]_j,$$
$$j = 2, 3 \ \Rightarrow \ [\mathcal{A}\Phi(t, r_*, s)]_j = [iV(s)\Phi(s - r_* + t, s)]_j.$$

¿From (171), for $r_* \leq R$, we deduce that

$$\begin{aligned}
\|\Phi(., r_*)\|_{H^1(]-\infty, R-r_*])^4} &\leq \left\| W^-_{0,\mathbb{R}} f \right\|_{H^1_{\mathbb{R}}} + 2\int_{-\infty}^{r_*} |V(s)| \, \|\Phi(r_* - s + ., r_*)\|_{H^1(]-\infty, R-r_*])^4} \, ds \\
&\quad + 2\int_{-\infty}^{r_*} |V(s)| \, \|\Phi(s - r_* + ., r_*)\|_{H^1(]-\infty, R-r_*])^4} \, ds, \\
&\leq \left\| W^-_{0,\mathbb{R}} f \right\|_{H^1_{\mathbb{R}}} + 2\int_{-\infty}^{r_*} |V(s)| \, \|\Phi(., r_*)\|_{H^1(]-\infty, R-s])^4} \, ds \\
&\quad + 2\int_{-\infty}^{r_*} |V(s)| \, \|\Phi(., r_*)\|_{H^1(]-\infty, R-2r_*+s])^4} \, ds, \\
&\leq C_1 \|f\|_{H^1_{\mathbb{R}}} + 4\int_{-\infty}^{r_*} |V(s)| \, \|\Phi(., r_*)\|_{H^1(]-\infty, R-s])^4} \, ds, \quad C_1 > 0.
\end{aligned}$$

Since, $V(s)$ is exponentially decreasing as $s \to -\infty$, by Gronwall lemma we obtain

$$\sup_{r_* \leq R} \|\Phi(., r_*)\|_{H^1(]-\infty, R-r_*])^4} < +\infty. \tag{172}$$

On the other hand, using (171) and (172), we have for $r_* \leq R$

$$\begin{aligned}
\left\| \Phi(., r_*) - W^-_{0,\mathbb{R}} f(r_* - .) \right\|_{H^1(]-\infty, R-r_*])^4} &\leq 4\int_{-\infty}^{r_*} |V(s)| \, \|\Phi(., r_*)\|_{H^1(]-\infty, R-s])^4} \, ds, \\
&\leq C_2 \int_{-\infty}^{r_*} |V(s)| ds, \quad C_2 > 0.
\end{aligned}$$

Thanks to the Sobolev embedding, for $r_* \leq R$, we conclude that

$$\sup_{\sigma \leq R - r_*} \left| \Phi(\sigma, r_*) - W^-_{0,\mathbb{R}} f(r_* - \sigma) \right| \leq C_3 \int_{-\infty}^{r_*} |V(s)| ds, \quad C_3 > 0. \tag{173}$$

We define

$$I := \int_0^{+\infty} \left| g_{\mathrm{T}}(t) - g^{\frac{T}{2}}(t) \right|^2 dt$$

and remark that

$$g^{\frac{T}{2}}(t) = \left[U_{0,\mathbb{R}}(t-T) \left(W_{0,\mathbb{R}}^- \right) \right] (-2t+1),$$
$$g_{\mathrm{T}}(t) = {}^t\!\left(0, \left[U_{V,\mathbb{R}}(t-T)f\right]_2, \left[U_{V,\mathbb{R}}(t-T)f\right]_3, 0\right)(-t+1).$$

Therefore, choosing $\chi \in C^\infty(\mathbb{R}_{r_*})$ a cut-off function such that

$$\exists\ a, b \in \mathbb{R}, \quad -\infty < a < b < 0 \quad \chi(r_*) = \left\{ \begin{array}{ll} 1 & r_* < a \\ 0 & r_* > b \end{array} \right. ,$$

and for $\zeta > 0$ we deduce that

$$\begin{aligned} I &\leq \int_0^{+\infty} \left| U_{0,\mathbb{R}}(t-T)\left(W_{0,\mathbb{R}}^- f\right)(-t+1) - \chi(-t+1) U_{V,\mathbb{R}}(t-T) f(-t+1) \right|^2 dt, \\ &\leq \zeta \sup_{\sigma \leq \zeta - T} \left\| U_{0,\mathbb{R}}(\sigma)\left(W_{0,\mathbb{R}}^- f\right) - \chi U_{V,\mathbb{R}}(\sigma) f \right\|^2_{L^\infty(\mathbb{R})^4} \\ &\quad + \int_\zeta^{+\infty} \left| U_{0,\mathbb{R}}(t-T)\left(W_{0,\mathbb{R}}^- f\right)(-t+1) - \Phi(t-T, -t+1) \right|^2 dt. \end{aligned}$$

By the Sobolev embedding and formula (173), for $\zeta, T \geq 1 - R$, we obtain

$$\begin{aligned} I &\leq \zeta \sup_{\sigma \leq \zeta - T} \left\| U_{0,\mathbb{R}}(\sigma)\left(W_{0,\mathbb{R}}^- f\right) - \chi U_{V,\mathbb{R}}(\sigma) f \right\|^2_{H^1_{\mathbb{R}}} \\ &\quad + \int_\zeta^{+\infty} \sup_{\sigma \leq t-T} \left| U_{0,\mathbb{R}}(\sigma)\left(W_{0,\mathbb{R}}^- f\right)(-t+1) - \Phi(\sigma, -t+1) \right|^2 dt, \quad T \geq R-1, \\ &\leq \zeta \sup_{\sigma \leq \zeta - T} \left\| U_{0,\mathbb{R}}(\sigma)\left(W_{0,\mathbb{R}}^- f\right) - \chi U_{V,\mathbb{R}}(\sigma) f \right\|^2_{H^1_{\mathbb{R}}} + C_3 \int_\zeta^{+\infty} \left(\int_{-\infty}^{-t+1} |V(s)| ds \right)^2 dt. \end{aligned}$$

Thanks to lemma 6.3 and since $V(s)$ is exponentially decreasing as $s \to -\infty$, we conclude that $\lim_{T \to +\infty} I = 0$.

■

Lemma 6.10
Given $f \in L^2_{\mathbb{R}}$, then

$$\lim_{T \to +\infty} \left\| \mathbf{1}_{[\delta, +\infty[}(D_{V,0}) \mathcal{J} U_V(0,T) f \right\|_0^2 = < W_{0,\mathbb{R}}^- f, e^{\frac{2\pi}{\kappa_0} D_{0,\mathbb{R}}} \left(1 + e^{\frac{2\pi}{\kappa_0} D_{0,\mathbb{R}}} \right)^{-1} W_{0,\mathbb{R}}^- f >_{L^2_{\mathbb{R}}}, \tag{174}$$

with

$$\delta = \frac{qQ}{r_0},$$

and

$$\mathcal{J} U_V(0,T) f \rightharpoonup 0, \quad T \to +\infty, \quad in \quad L^2_0. \tag{175}$$

Proof:
For $f \in \mathcal{D}_{\mathrm{R}}$, $R > 0$ fixed, thanks to (167), (168), (124) and (114), we have

$$\begin{aligned} \left\| \mathcal{J} U_V(0,T) f - \left[G_V\left(g^{\frac{T}{2}}\right) \right]_L \right\|_0^2 &= \left\| \left[G_V(g_{\mathrm{T}}) \right]_L - \left[G_V\left(g^{\frac{T}{2}}\right) \right]_L \right\|_0^2, \\ &\leq 2 \int_0^{+\infty} \left| g_{\mathrm{T}}(t) - g^{\frac{T}{2}}(t) \right|^2 dt \to 0, \quad T \to +\infty. \end{aligned} \tag{176}$$

According to lemma 6.9

$$\begin{aligned}
\lim_{T\to+\infty} & \left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\right\|_0^2 \\
&= \lim_{T\to+\infty}\left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\left[G_V\left(g^{\frac{T}{2}}\right)\right]_L\right\|_0^2, \\
&= 2\int_{\mathbb{R}} < W_{0,\mathbb{R}}^- f(1-2t), e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} W_{0,\mathbb{R}}^- f(1-2t) >_{\mathbb{C}^4} dt, \\
&=< W_{0,\mathbb{R}}^- f, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} W_{0,\mathbb{R}}^- f >_{L^2_{\mathbb{R}}}.
\end{aligned}$$

With limit (176) and lemma 6.7 we obtain (175) for $f \in \mathcal{D}_{\mathrm{R}}$. Since all norms are uniformly bounded with respect to T, lemma is proved by density. ■

Finally, we prove the main result of this subpart:

Proposition 6.4
Given $f \in L^2_{\mathbb{R}}$, *then*

$$\begin{aligned}
\lim_{T\to+\infty}\left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})U_V(0,T)f\right\|_0^2 &= \left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})W^-_{V,[z(0),+\infty[}f\right\|_0^2 \\
&+ < W_{0,\mathbb{R}}^- f, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} W_{0,\mathbb{R}}^- f >_{L^2_{\mathbb{R}}}, \qquad (177)
\end{aligned}$$

with

$$\delta = \frac{qQ}{r_0}.$$

Proof:
With simple calculation, we deduce

$$\begin{aligned}
\left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})U_V(0,T)f\right\|_0^2 &= \left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f\right\|_0^2 + \left\|\mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T)f\right\|_0^2 \\
&+ 2\Re < \mathbf{1}_{[\delta,+\infty[}(D_{V,0})(1-\mathcal{J})U_V(0,T)f, \mathbf{1}_{[\delta,+\infty[}(D_{V,0})\mathcal{J}U_V(0,T)f >_{L^2_0}.
\end{aligned}$$

According to limit (175) and lemma 6.3 the last term is zero as $T \to +\infty$. The two norms are by lemma 6.10 and lemma 6.3. ■

6.3.2 Proof of theorem 5.2

Now, we prove the key estimate. Using operators (69), (71) and the properties (73), (74) and (87), by Lebesgue theorem and proposition 6.4, we have

$$\begin{aligned}
\left\|\mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_0)\,\boldsymbol{U}(0,T)f\right\|_0^2 &= \sum_{(l,n)\in\mathcal{I}}\left\|\mathbf{1}_{[0,+\infty[}(D_{V_{l,\nu},0}-\delta)U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f\right\|_0^2, \\
&= \sum_{(l,n)\in\mathcal{I}}\left\|\mathbf{1}_{[\delta,+\infty[}(D_{V_{l,\nu},0})U_{V_{l,\nu}}(0,T)\mathcal{R}^\nu_{ln}f\right\|_0^2, \\
\underset{T\to+\infty}{\longrightarrow} & \sum_{(l,n)\in\mathcal{I}}\left\|\mathbf{1}_{[\delta,+\infty[}(D_{V_{l,\nu},0})W^-_{V_{l,\nu},[z(0),+\infty[}\mathcal{R}^\nu_{ln}f\right\|_0^2 \\
&+ \sum_{(l,n)\in\mathcal{I}} < W_{0,\mathbb{R}}^-\mathcal{R}^\nu_{ln}f, e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\left(1+e^{\frac{2\pi}{\kappa_0}D_{0,\mathbb{R}}}\right)^{-1} W_{0,\mathbb{R}}^-\mathcal{R}^\nu_{ln}f >_{L^2_{\mathbb{R}}}, \\
&=: S_1 + S_2.
\end{aligned}$$

By lemma 6.3, the wave operator $W^-_{V_{l,\nu},[z(0),+\infty[}$ exists and is an isometry from $L^2_{\mathbb{R}}$ onto $P_{ac}(D_{V,[z(0),+\infty[})L^2_0$. Hence

$$\boldsymbol{W}^-_+ := \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} W^-_{V_{l,\nu},[z(0),+\infty[} \mathcal{R}^\nu_{ln}$$

exists and is an isometry from $\boldsymbol{L}^2_{\text{BH}}$ onto $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$. We put

$$\boldsymbol{\Omega}^-_{\Lambda,\rightarrow} := \left(\boldsymbol{W}^-_{\Lambda,\rightarrow}\right)^*, \quad \left(\text{resp.} \quad \boldsymbol{\Omega}^-_{0,\rightarrow} := \left(\boldsymbol{W}^-_{0,\rightarrow}\right)^*\right).$$

If $\Lambda > 0$ (resp. $\Lambda = 0$), we define the wave operator:

$$\boldsymbol{W}^-_{\Lambda,D} : P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0 \to \boldsymbol{L}^2_{\Lambda,\rightarrow}, \quad \left(\text{resp.} \quad \boldsymbol{W}^-_{0,D} : P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0 \to \boldsymbol{L}^2_{0,\rightarrow}\right),$$

such that

$$\boldsymbol{W}^-_{\Lambda,D} := \boldsymbol{\Omega}^-_{\Lambda,\rightarrow}\left(\boldsymbol{W}^-_+\right)^*, \quad \left(\text{resp.} \quad \boldsymbol{W}^-_{0,D} := \boldsymbol{\Omega}^-_{0,\rightarrow}\left(\boldsymbol{W}^-_+\right)^*\right).$$

By the chain rule theorem, these operators are isometries from $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$ onto $\boldsymbol{L}^2_{\Lambda,\rightarrow}$, (resp. $P_{ac}(\boldsymbol{D}_0)\boldsymbol{L}^2_0$ onto $\boldsymbol{L}^2_{0,\rightarrow}$). From the previous discussion and the intertwining properties, we have, if $\Lambda \geq 0$

$$\begin{aligned}
S_1 &= \sum_{(l,n)\in\mathcal{I}} \left\| W^-_{V_{l,\nu},[z(0),+\infty[} \mathbf{1}_{[\delta,+\infty[}(D_{V,\mathbb{R}})\mathcal{R}^\nu_{ln} f \right\|^2_0, \\
&= \left\| \boldsymbol{W}^-_+ \mathbf{1}_{[\delta,+\infty[}(\boldsymbol{D}_{\text{BH}} + \delta) f \right\|^2_0, \\
&= \left\| \boldsymbol{W}^-_{\Lambda,D}\, \boldsymbol{W}^-_+ \mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\text{BH}}) f \right\|^2_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \\
&= \left\| \boldsymbol{\Omega}^-_{\Lambda,\rightarrow} \mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\text{BH}}) f \right\|^2_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}, \\
&= \left\| \mathbf{1}_{[0,+\infty[}(\boldsymbol{D}_{\Lambda,\rightarrow}) \boldsymbol{\Omega}^-_{\Lambda,\rightarrow} f \right\|^2_{\boldsymbol{L}^2_{\Lambda,\rightarrow}}.
\end{aligned}$$

We put

$$\boldsymbol{\Omega}^-_{\leftarrow} := \left(\boldsymbol{W}^-_{\leftarrow}\right)^*,$$

and remark that

$$\mathcal{P}_r \boldsymbol{D}_{\leftarrow} \mathcal{P}_r^{-1} = \bigoplus_{(l,n)\in\mathcal{I}} \mathcal{E}^\nu_{ln} D_{0,\mathbb{R}} \mathcal{R}^\nu_{ln} - \delta, \quad \delta = \frac{qQ}{r_0}.$$

Then, with (73) and (112), we obtain that

$$\begin{aligned}
S_2 &= < \mathcal{P}_r \boldsymbol{\Omega}^-_{\leftarrow} f, \mathcal{P}_r \mu e^{\frac{2\pi}{\kappa_0}\boldsymbol{D}_{\leftarrow}} \left(1 + \mu e^{\frac{2\pi}{\kappa_0}\boldsymbol{D}_{\leftarrow}}\right) \boldsymbol{\Omega}^-_{\leftarrow} f >_{\boldsymbol{L}^2_{\text{BH}}}, \quad \boldsymbol{L}^2_{\text{BH}} = \mathcal{P}_r \boldsymbol{L}^2_{\leftarrow}, \\
&= < \boldsymbol{\Omega}^-_{\leftarrow} f, \mu e^{\sigma \boldsymbol{D}_{\leftarrow}} \left(1 + \mu e^{\sigma \boldsymbol{D}_{\leftarrow}}\right)^{-1} \boldsymbol{\Omega}^-_{\leftarrow} f >_{\boldsymbol{L}^2_{\leftarrow}}, \quad \mu = e^{\sigma\delta}, \quad \sigma := \frac{2\pi}{\kappa_0}, \quad \delta := \frac{qQ}{r_0}.
\end{aligned}$$

involving the limits (58). ■

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References

[1] A. Bachelot, *Quantum vacuum polarization at the black-hole horizon*, Ann. Inst. H. Poincaré Phys. Théor. **67** (1997), no. 2, 181–222.

[2] ———, *Scattering of scalar fields by spherical gravitational collapse*, J. Math. Pures Appl. (9) **76** (1997), no. 2, 155–210.

[3] ———, *The Hawking effect*, Ann. Inst. H. Poincaré Phys. Théor. **70** (1999), no. 1, 41–99.

[4] ———, *Creation of fermions at the charged black-hole horizon*, Ann. Henri Poincaré **1** (2000), no. 6, 1043–1095.

[5] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge, 1982.

[6] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, *New extended model of hadrons*, Phys. Rev. D (3) **9** (1974), no. 12, 3471–3495.

[7] J. Dereziński and C. Gérard, *Scattering theory of classical and quantum N-particle systems*, Springer-Verlag, Berlin, 1997.

[8] J. Dimock, *Algebras of local observables on a manifold*, Comm. Math. Phys. **77** (1980), no. 3, 219–228.

[9] ———, *Dirac quantum fields on a manifold*, Trans. Amer. Math. Soc. **269** (1982), no. 1, 133–147.

[10] J. Dimock and B. S. Kay, *Classical and quantum scattering theory for linear scalar fields on the Schwarzschild metric. I*, Ann. Physics **175** (1987), no. 2, 366–426.

[11] K. Fredenhagen and R. Haag, *On the derivation of Hawking radiation associated with the formation of a black hole*, Comm. Math. Phys. **127** (1990), no. 2, 273–284.

[12] I. M. Gel′fand and Z. Ya. Šapiro, *Representations of the group of rotations of 3-dimensional space and their applications*, Amer. Math. Soc. Transl. (2) **2** (1956), 207–316.

[13] G. W. Gibbons and S. W. Hawking, *Cosmological event horizons, thermodynamics, and particle creation*, Phys. Rev. D (3) **15** (1977), no. 10, 2738–2751.

[14] S. W. Hawking, *Particle creation by black holes*, Comm. Math. Phys. **43** (1975), no. 3, 199–220.

[15] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, London, 1973, Cambridge Monographs on Mathematical Physics, No. 1.

[16] W. M. Jin, *Scattering of massive Dirac fields on the Schwarzschild black hole spacetime*, Classical Quantum Gravity **15** (1998), no. 10, 3163–3175.

[17] F. Melnyk, *Wave operators for the massive charged linear Dirac field on the Reissner-Nordström metric*, Classical Quantum Gravity **17** (2000), no. 11, 2281–2296.

[18] ———, *Diffusion d'un champ massif et chargé de Dirac en métrique de Reissner-Nordstrøm*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 3, 185–190.

[19] ———, *Scattering on Reissner-Nordstrøm metric for the massive charged spin 1/2 fields*, Ann. Henri Poincaré (to appear) (2003).

[20] C. W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, W. H. Freeman and Co., San Francisco, Calif., 1973.

[21] J.-P. Nicolas, *Scattering of linear Dirac fields by a spherically symmetric black hole*, Ann. Inst. H. Poincaré Phys. Théor. **62** (1995), no. 2, 145–179.

[22] ______, *Global exterior Cauchy problem for spin* 3/2 *zero rest-mass fields in the Schwarzschild spacetime*, Comm. Partial Differential Equations **22** (1997), no. 3-4, 465–502.

[23] M. Reed and B. Simon, *Methods of modern mathematical physics. I, II, III, IV*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1972-75-78-79.

[24] B. Thaller, *The Dirac equation*, Springer-Verlag, Berlin, 1992.

[25] R. M. Wald, *On particle creation by black holes*, Comm. Math. Phys. **45** (1975), no. 1, 9–34.

[26] ______, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, Chicago, IL, 1994.